GENERAL EQUILIBRIUM WITH THE STANDARD INCOMPLETE MARKETS MODEL

Econ 411-3

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RECALLING PARTIAL EQUILIBRIUM PROBLEM (STEADY STATE)

- Steady-state equilibrium consisted of:
  - **policy functions** $a'(e,a)$ and $c(e,a)$ that solve Bellman
  - **measure** $\mu(e,\mathbb{A})$ that satisfies steady-state law of motion
  - together implying aggregate assets and consumption:

$$A = \int a d\mu = \int a'(e,a) d\mu \qquad C = \int c(e,a) d\mu$$

We'll endogenize these as returns to capital and labor in general equilibrium!

- Problem has various parameters:
  - household discount rate β and utility u
  - borrowing constraint $\underline{a}$, Markov process on e
  - real rate r, incomes $y(e)$

ADDING A PRODUCTION SIDE TO THE MODEL

- Suppose competitive CRTS firm uses capital and labor

$$Y_t = F(K_{t-1}, L_t)$$

- Law of motion for capital is

$$K_t = I_t + (1 - \delta)K_{t-1}$$

- Goods can be spent on consumption or investment

$$Y_t = C_t + I_t$$

- One unit of investment at t-1 gives net marginal return at t of

$$F_K(K_{t-1}, L_t) - \delta$$

- Equate in steady state with real interest rate r to obtain

$$F_K(K, L) = r + \delta$$

LABOR SIDE AND GENERAL EQUILIBRIUM

- Marginal return of hiring labor equated with wage

$$F_L(K, L) = w$$

- Assume each household has stochastic, inelastic endowment of labor e, which it supplies at wage w, delivering income

$$y(e) = we$$

- Total labor supply, equaling demand in equilibrium, is just

$$L = \int e d\mu(e, a)$$

- Also suppose household assets are claims to capital, so

$$K = A = \int a d\mu(e, a)$$

SUMMING UP (STEADY STATE) GENERAL EQUILIBRIUM

- General equilibrium is partial equilibrium of standard incomplete markets model, with additionally:

Since we assume process for e is exogenous, L is exogenous, can treat as just a fixed number

Walras's law implies that this goods market clearing condition is redundant gives asset market clearing and other conditions

$$y(e) = we$$

$$F_L(K, L) = w$$

$$F_K(K, L) = r + \delta$$

$$K = A = \int a d\mu(e, a)$$

$$L = \int e d\mu(e, a)$$

$$Y = I + C = \delta K + \int c(e, a) d\mu(e, a)$$

EXERCISE: LET'S SHOW THAT GOODS CLEARING IS REDUNDANT

- Household period-by-period budget constraint:

$$a_{it} + c_{it} = (1+r)a_{i,t-1} + we_{it}$$

- Aggregated in steady state:

$$C = rA + wL$$

- Substitute $r = F_K - \delta$, $w = F_L$, and $A = K$:

$$C = F_K K + F_L L - \delta K$$

- Euler identity $Y = F_K K + F_L L$, and $I = \delta K$:

$$C + I = Y$$

Steady-state goods market clearing derived from other constraints!

WHAT IS THIS MODEL?

- This is called the **Aiyagari model**, after Aiyagari (1994)
  - treatments often jump straight to this general equilibrium setup, but I think it's better to see the partial equilibrium problem in isolation first

- One of three canonical general equilibrium setups:
  - **Aiyagari (1994)**: aggregate hh assets = **capital**
  - **Huggett (1993)**: aggregate hh assets = **zero**
  - **Bewley (1980, 1983)**: aggregate hh assets = **gov bonds**

Actually he called it "money", but interpretation today would usually be government bonds

ANALYTICS OF THE AIYAGARI MODEL

SIMPLIFYING OBSERVATIONS 1: HOUSEHOLD SIDE

- Holding other parameters constant, steady-state A is a function of rates r and wages w determined in GE

$$A(r, w)$$

- But under two assumptions—CRRA utility u and a borrowing constraint of zero—it actually scales linearly in w, so can write for some a(r)

$$A(r, w) = a(r)wL$$

- Intuition: if w scales up, scale all policies with it, everything still optimal [formal proof in problem set?]

SIMPLIFYING OBSERVATIONS 2: PRODUCTION SIDE

- Assuming F CRTS, continuous and strictly concave, write

$$f(k) \equiv F(k,1)$$

- This is also strictly concave, and

$$f'(K/L) = F_K(K/L,1) = F_K(K,L) = r + \delta$$

- Unique solution k=K/L satisfying this, define implicitly

$$f'(k(r)) \equiv r + \delta$$

as capital-labor ratio consistent with r (declining function)

SIMPLIFYING OBSERVATIONS 2: PRODUCTION SIDE CONTINUED

- We also have by Euler's identity

$$w = F_L(K,L) = \frac{F(K,L) - F_K(K,L)K}{L} = f(K/L) - f'(K/L)(K/L)$$

- Given $k(r)$ from last slide, can substitute in to get $w(r)$:

$$w(r) \equiv f(k(r)) - f'(k(r))k(r)$$

- Can show that this is strictly declining in r as well:

$$w'(r) = f'(k)k'(r) - f'(k)k'(r) - f''(k)k'(r) = -f''(k)k'(r) < 0$$

SUMMING UP WHAT WE KNOW

- Key equilibrium condition is A=K, which becomes

$$a(r)w(r)L = k(r)L$$

- Or, dividing by L and rearranging

$$a(r) = \frac{k(r)}{w(r)}$$

- This is a simple equation in r:
  - left likely increasing in r (based on examples we've seen)
  - right likely decreasing in r (if numerator dominates denom)

SUMMING UP WHAT WE KNOW

- Key equilibrium condition is A=K, which becomes

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Steady-state aggregate ***asset demand*** *relative to labor income*

$$a(r) = \frac{k(r)}{w(r)}$$

Steady-state aggregate ***asset supply*** *relative to labor income*

- This is a simple equation in r:
  - left likely increasing in r (based on examples we've seen)
  - right likely decreasing in r (if numerator dominates denom)

Everything follows from equilibrium r!

SPECIAL CASE: COBB-DOUGLAS TECHNOLOGY

- Assume that

$$Y = F(K, L) = K^{\alpha} L^{1-\alpha}$$

- Then

$$r + \delta = F_K = \alpha \frac{Y}{K} \qquad w = F_L = (1-\alpha)\frac{Y}{L}$$

- So

$$\frac{w}{r+\delta} = \frac{1-\alpha}{\alpha}\frac{K}{L} \Rightarrow \frac{k(r)}{w(r)} = \frac{\alpha}{1-\alpha}\frac{1}{r+\delta}$$

- Closed-form solution for asset supply vs. labor income, unambiguously decreasing!

MORE GENERAL SETTING: ANYTHING GOES

- It is exceedingly hard to find examples where a(r) is not strictly increasing (at least when above 0), but it is possible
  - Intuition: for counterexample need **income effects** to dominate **substitution effects**, so “I need to save less because I’ll earn more on my savings” dominates “I want to save more because I’ll earn more on my savings”
  - Easier to see cases in related life-cycle setting (saving for retirement)
- Easier to find cases where k(r)/w(r) is sometimes increasing
  - Need decreasing denominator to dominate decreasing numerator, possible when **elasticity of substitution is low** (lower than 1) but capital share high
- We’ll generally assume a(r) increasing, k(r)/w(r) decreasing, so unique solution

EFFECT OF A FIRST-ORDER SHOCK

- Add extra parameter θ that represents some arbitrary parameter in the steady-state household problem
  - Could be discounting rate, constraint, income risk, etc.

- Take logs and apply implicit function theorem:

$$a(r,\theta) = \frac{k(r)}{w(r)} \rightarrow \log a(r,\theta) = \log k(r) - \log w(r)$$

$$\epsilon^d_\theta \equiv \partial \log a / \partial \theta$$

$$\epsilon^d_r \equiv \partial \log a / \partial r$$

$$\epsilon^s_r \equiv -(\partial \log k / \partial r - \partial \log w / \partial r)$$

Semielasticities of asset demand and supply

$$dr = -\frac{\epsilon^d_\theta d\theta}{\epsilon^d_r + \epsilon^s_r}$$

UNDERSTANDING THE ANALYTICS

$$dr = -\frac{\epsilon_\theta^d d\theta}{\epsilon_r^d + \epsilon_r^s}$$

$\epsilon_\theta^d d\theta$ is the PE effect of shift on log asset demand

The necessary equilibrating decline in r is inversely proportional to the sum of semielasticities, since this will cause a decrease in log asset demand of $(\epsilon_r^d + \epsilon_r^s)dr$

HOW MUCH QUANTITY ADJUSTMENT HAPPENS?

- Can multiply dr by the semielasticity of k(r)/w(r) to get:

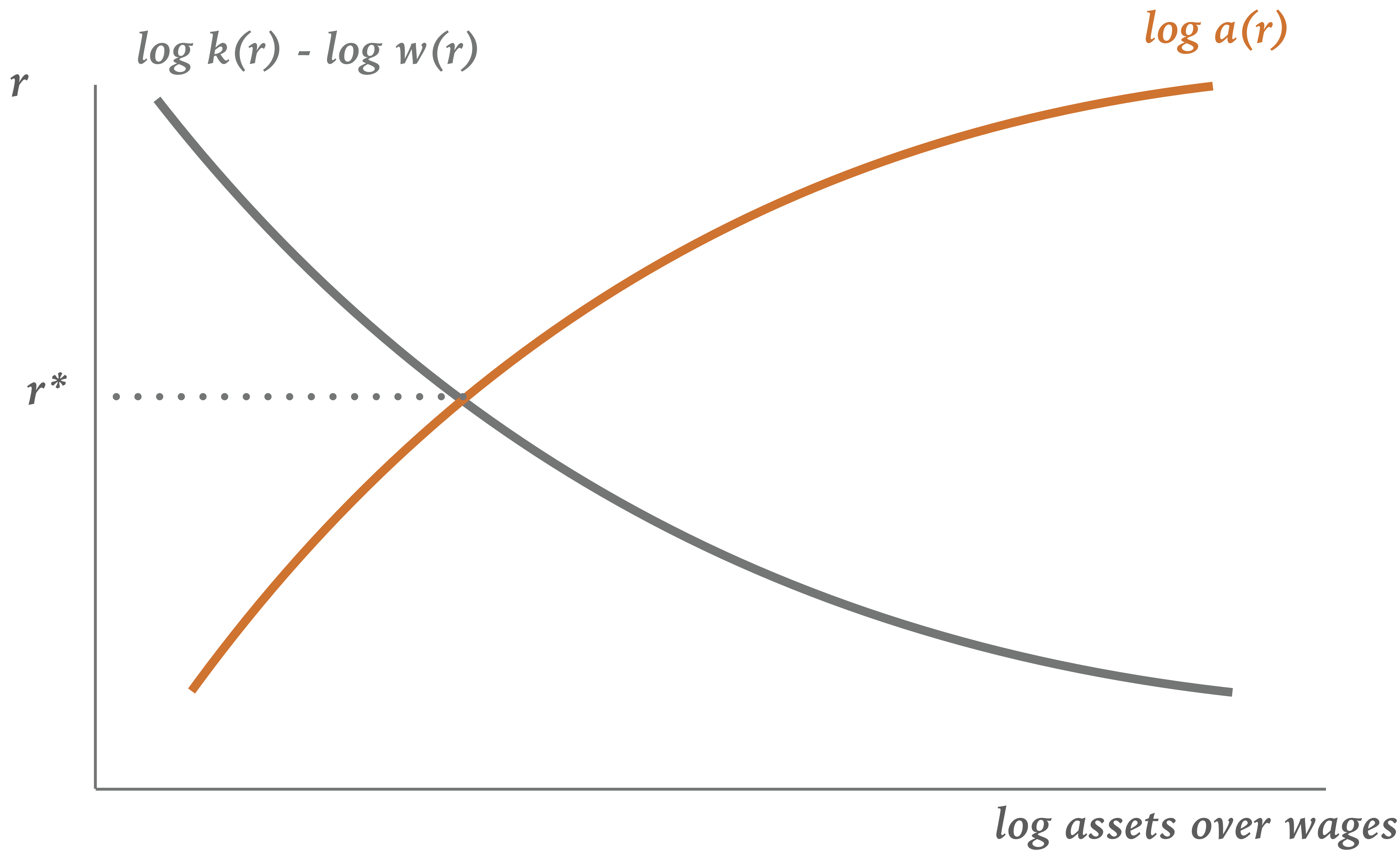
$$d\log(k(r)/w(r)) = -\epsilon_r^s dr = \frac{\epsilon_r^s}{\epsilon_r^d + \epsilon_r^s}\epsilon_\theta^d d\theta$$

- Fraction of the partial equilibrium impulse to asset demand that shows up in the capital-to-labor income ratio is proportional to supply's share of adjustment

$$\frac{\epsilon_r^s}{\epsilon_r^d + \epsilon_r^s}$$

- If Cobb-Douglas (our usual case), this is also the change in log(K/Y). Need to do a bit more work to get Y, or K/Y in other cases

GRAPHICALLY



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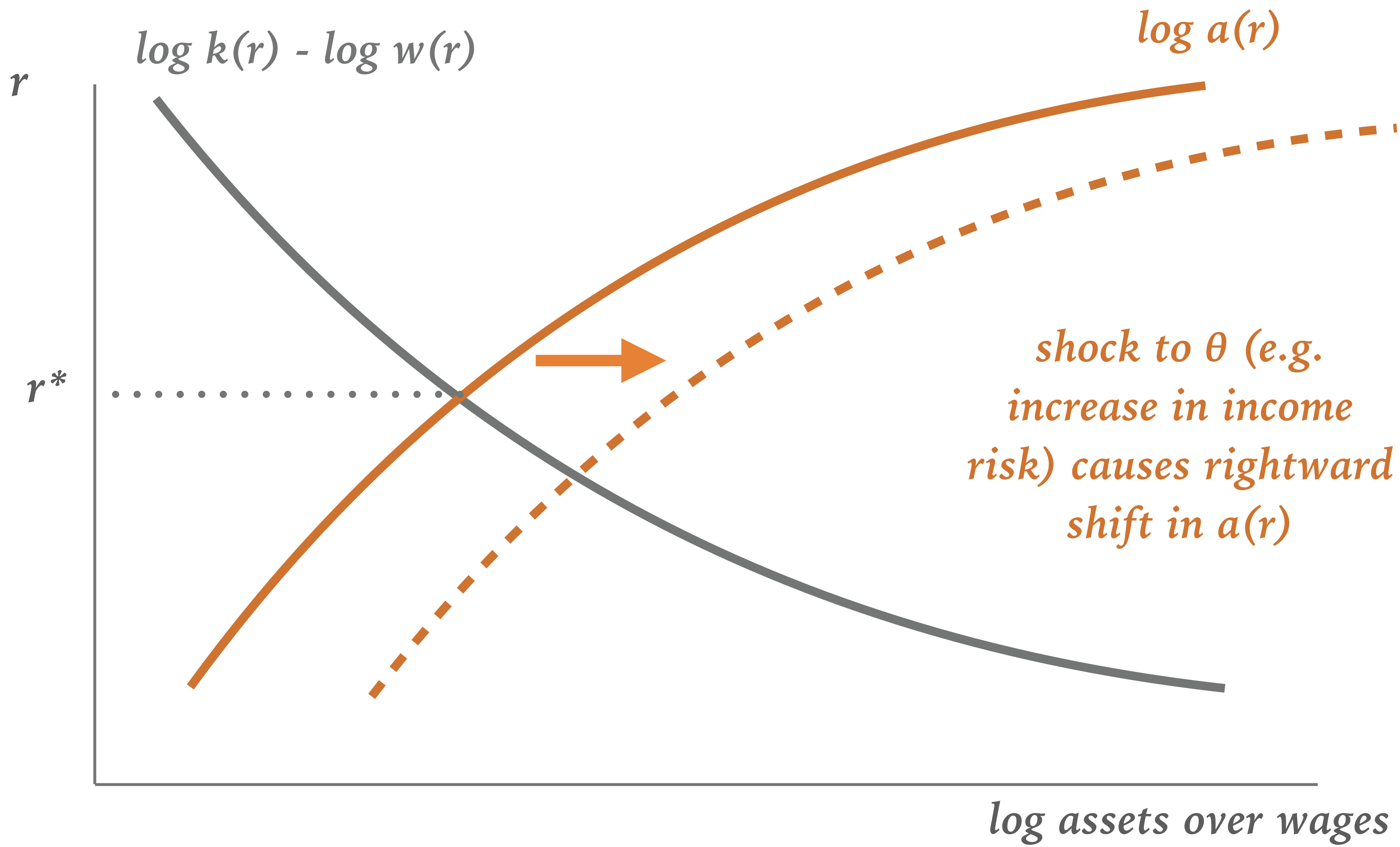

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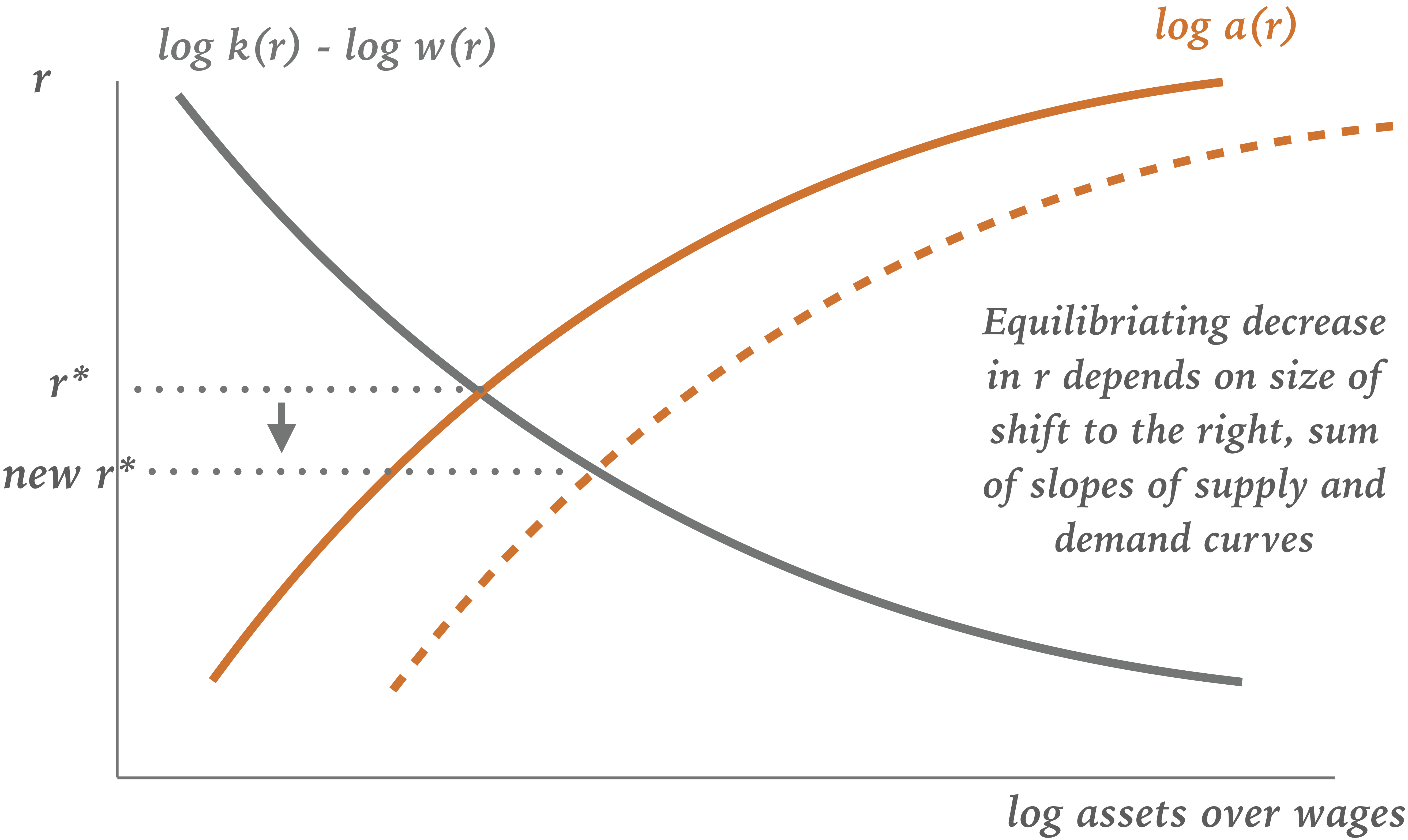

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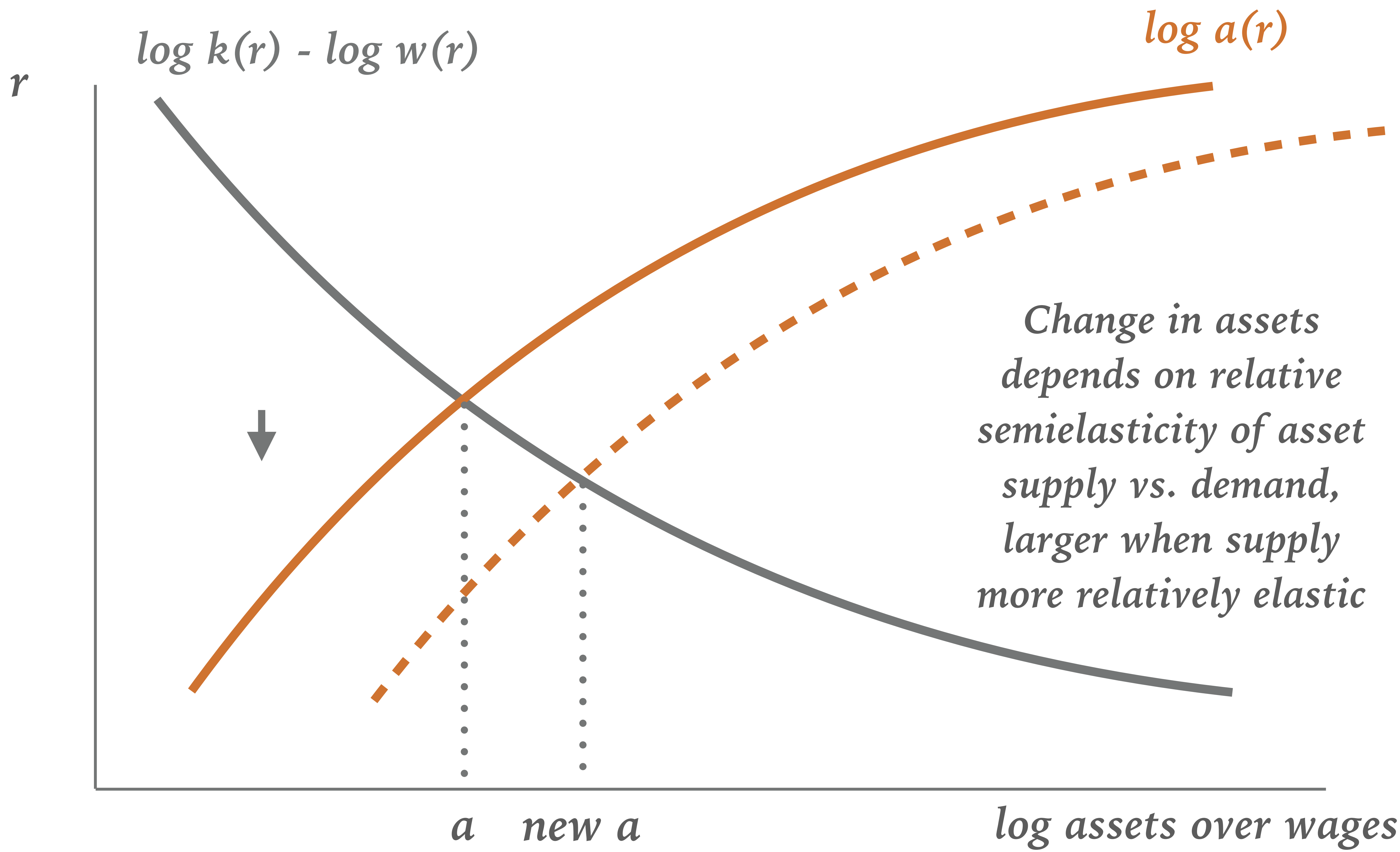


GENERAL EQUILIBRIUM IN PRACTICE

HOW DO WE CALIBRATE A MODEL?

- Household side is the same as before, swapping out y(e) for e, which we can calibrate the same way to match log income process (the average level of e will correspond to the L on the supply side)
- Now we have a production side too
- To **calibrate** one steady state: usually we will have some r and ratio of capital and labor income we are targeting, to match observed values, so we write this and solve for β:

$$a(r, \beta) = \frac{k(r)}{w(r)}$$

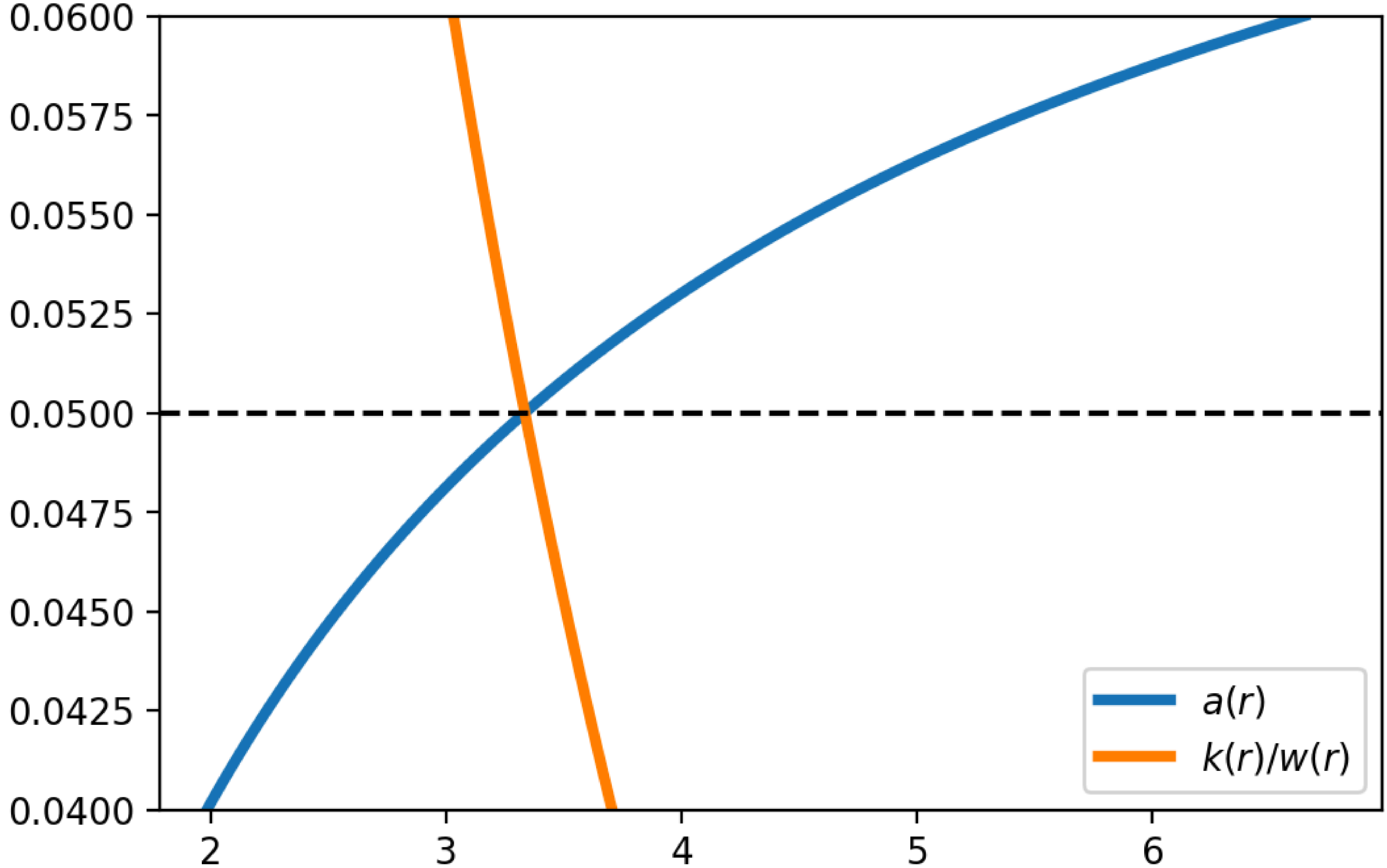
GOING BEYOND STEADY STATE

- Once we've fixed β and have full steady-state household side, might consider shocks to various things (income risk, etc.)
- Then we are looking for **equilibrating r**, like in our diagrams:

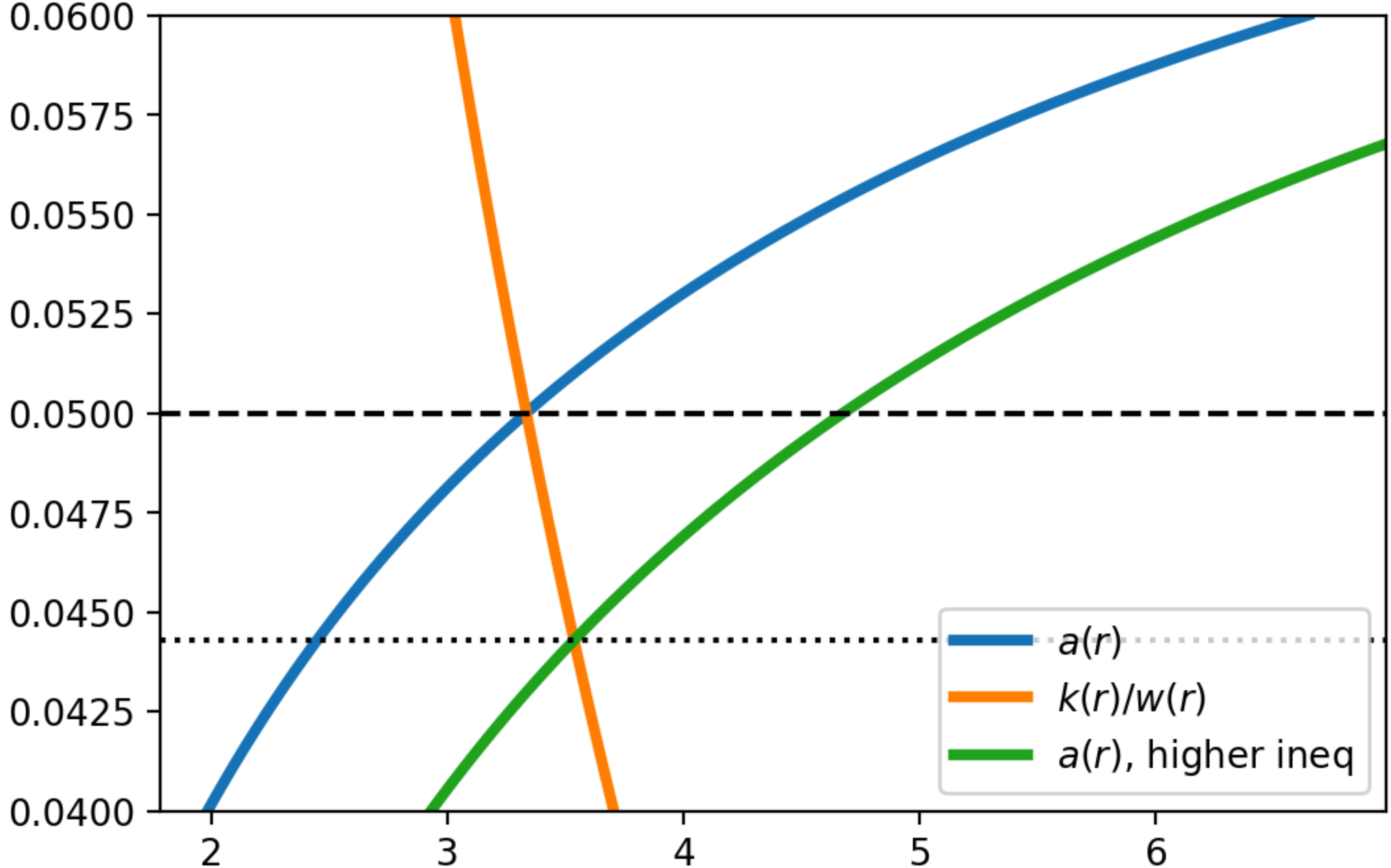
$$a(r, \theta) = \frac{k(r)}{w(r)}$$

- Need assumption that gives the shape of k(r)/w(r) vs. r, not just level, and then solve for r
  - we'll probably just use Cobb-Douglas, which is simplest
  - can solve for r in response to large shock nonlinearly, or use our first-order analytical formulas

CURVES IN OUR CALIBRATED MODEL (SEE JUPYTER NOTEBOOK)



INCREASE SD OF LOG INCOME BY 12 LOG POINTS (~RISE FROM 1980 TO NOW)



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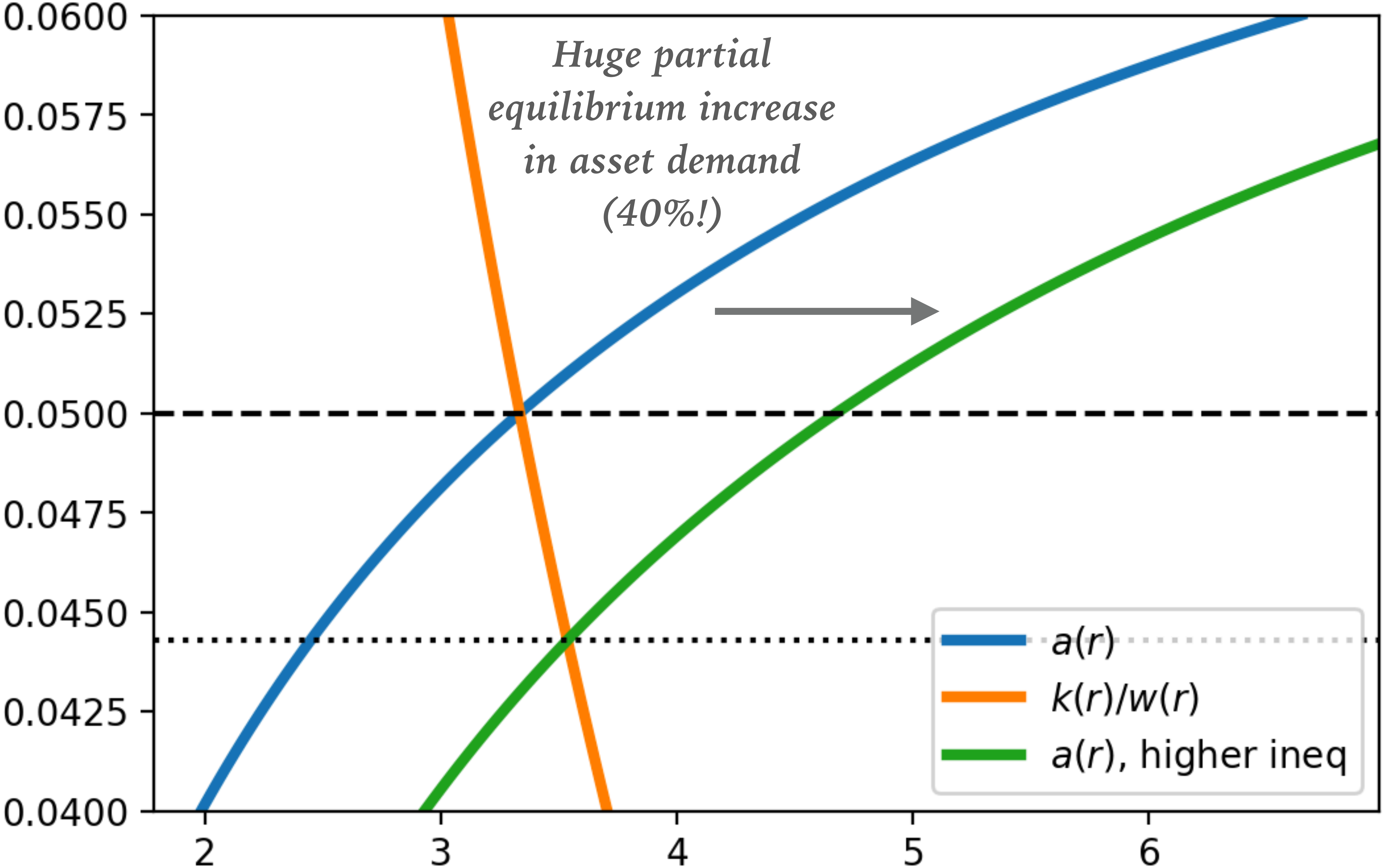

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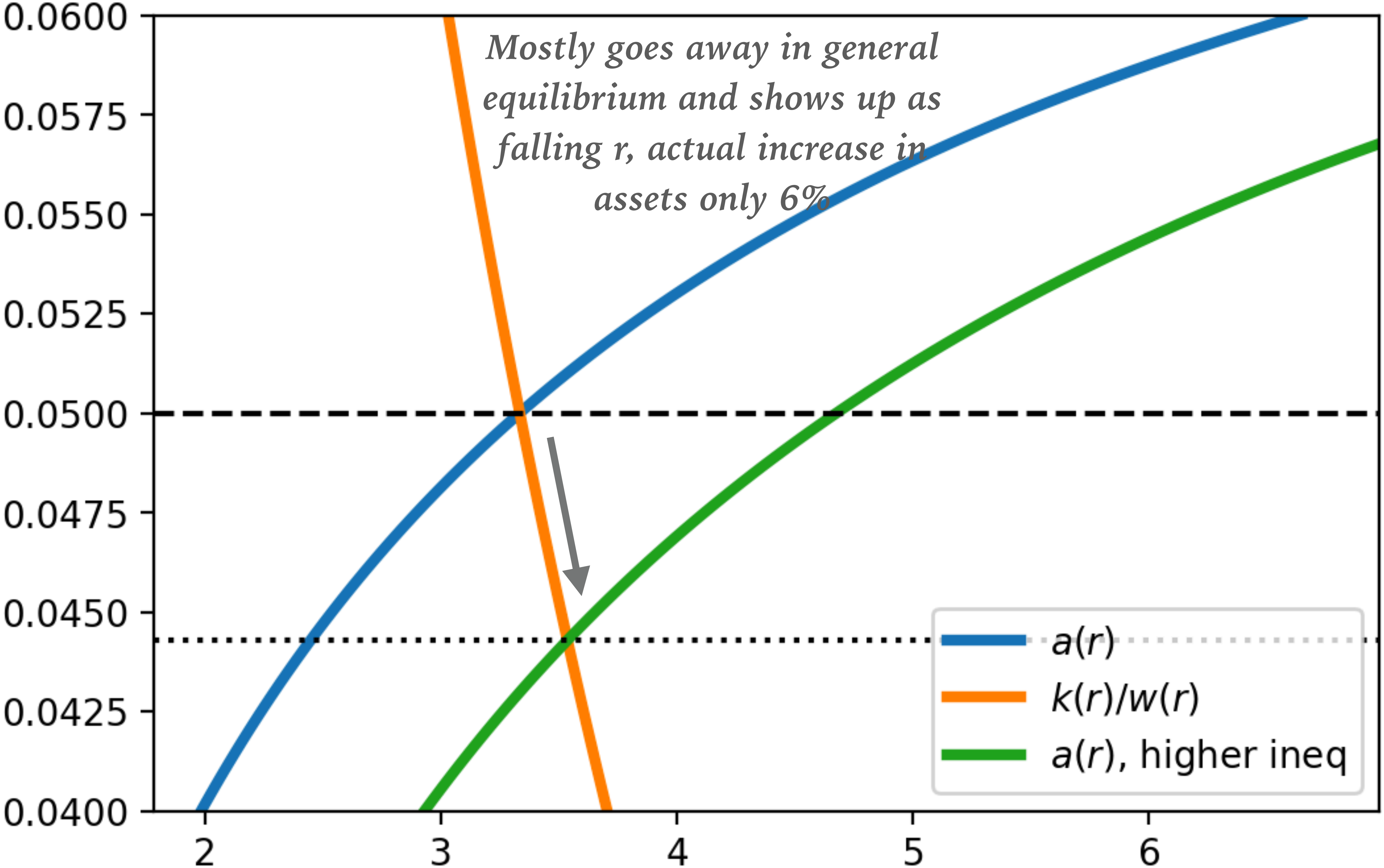


HOW WELL DID OUR APPROXIMATIONS DO (CONVERT TO ANNUAL)

$$dr = -\frac{\epsilon_\theta^d d\theta}{\epsilon_r^d + \epsilon_r^s}$$

$$\approx -0.5\,\%$$

$$\epsilon_\theta^d d\theta \approx 0.34$$

$$\epsilon_r^d \approx 58$$

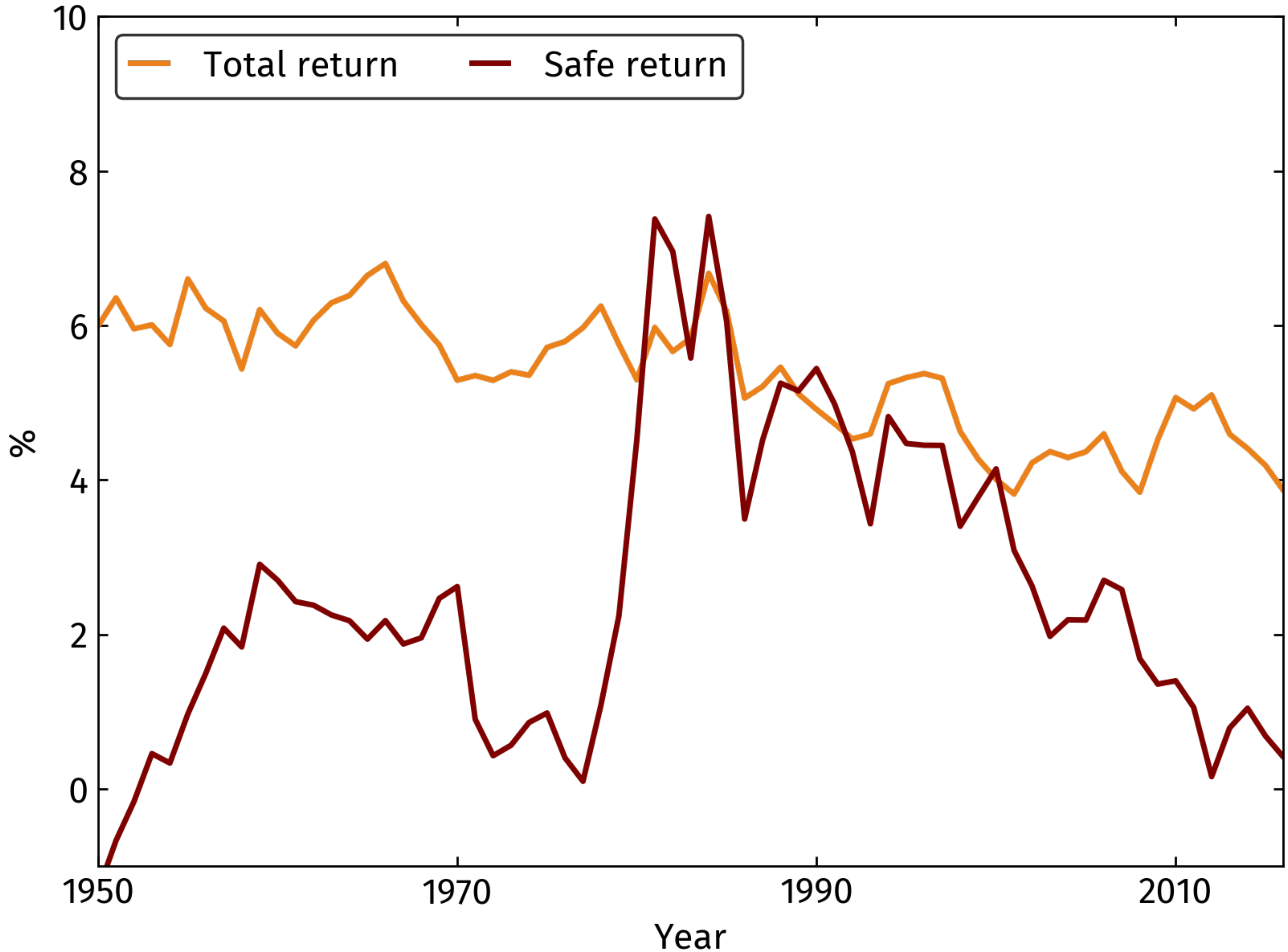
$$\epsilon_r^s \approx 10$$

Actual change is $\approx -0.57\,\%$

Why different? Because it's a big shock, and the semielasticity of asset demand falls with lower r enough to kick in and expand this effect. Share of PE adjustment that shows up in GE about 15%, very close to

$$\frac{\epsilon_r^s}{\epsilon_r^d + \epsilon_r^s} \approx 0.15$$

THIS KIND OF CHANGE COULD EXPLAIN HEALTHY FRACTION OF DECLINE IN R!



CAVEATS AND OTHER POSSIBLE SHOCKS

- By increasing inequality, we also increased **income risk** proportionally, which drove increased asset demand in model
  - if we just made some people permanently richer than others, would be **no effect** in this model (scaling)
  - Straub (2019): data suggests that permanent-income households do save more, so maybe this should still work

- Leading other accounts of falling r:
  - Aging population (e.g. Auclert Malmberg Martenet Rognlie)
  - Falling productivity growth