THE PARETO DISTRIBUTION AND FAT TAILS FOR INCOME AND WEALTH

Econ 411-3
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WHAT IS A PARETO DISTRIBUTION? SOME MATH

PARETO DISTRIBUTION: CDF AND PDF

$$F(x) = \begin{cases} 1 - \left(\frac{x_m}{x}\right)^\alpha & x \geq x_m \\ 0 & x < x_m \end{cases}$$

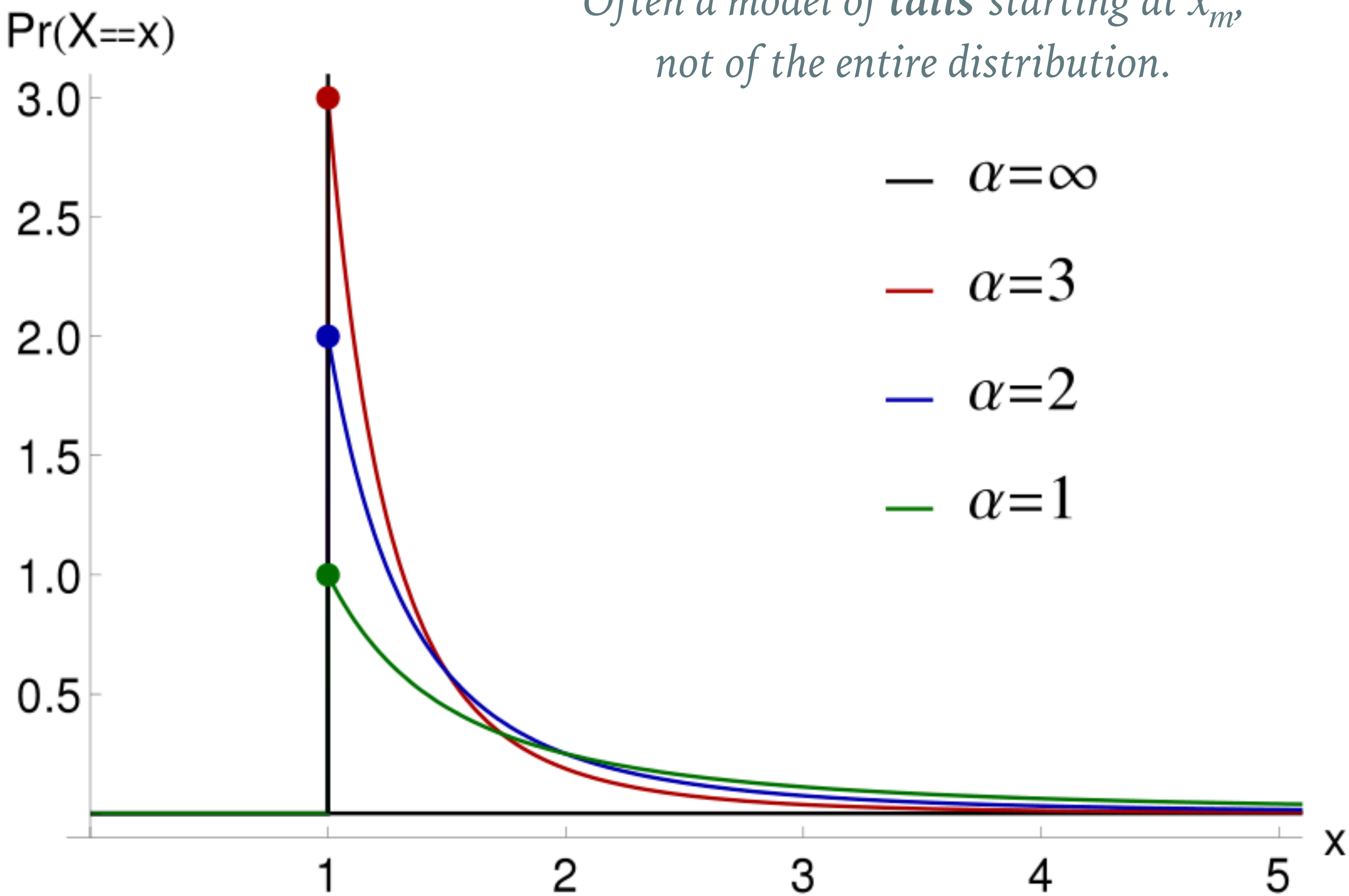
$$f(x) = \begin{cases} \frac{\alpha x_m^\alpha}{x^{\alpha+1}} & x \geq x_m \\ 0 & x < x_m \end{cases}$$

α *"Shape parameter"*

x_m *"Scale parameter" / minimum*

PARETO DISTRIBUTION, AN EXAMPLE VISUALIZED (WITH X_M=1)

*Often a model of **tails** starting at x_m, not of the entire distribution.*

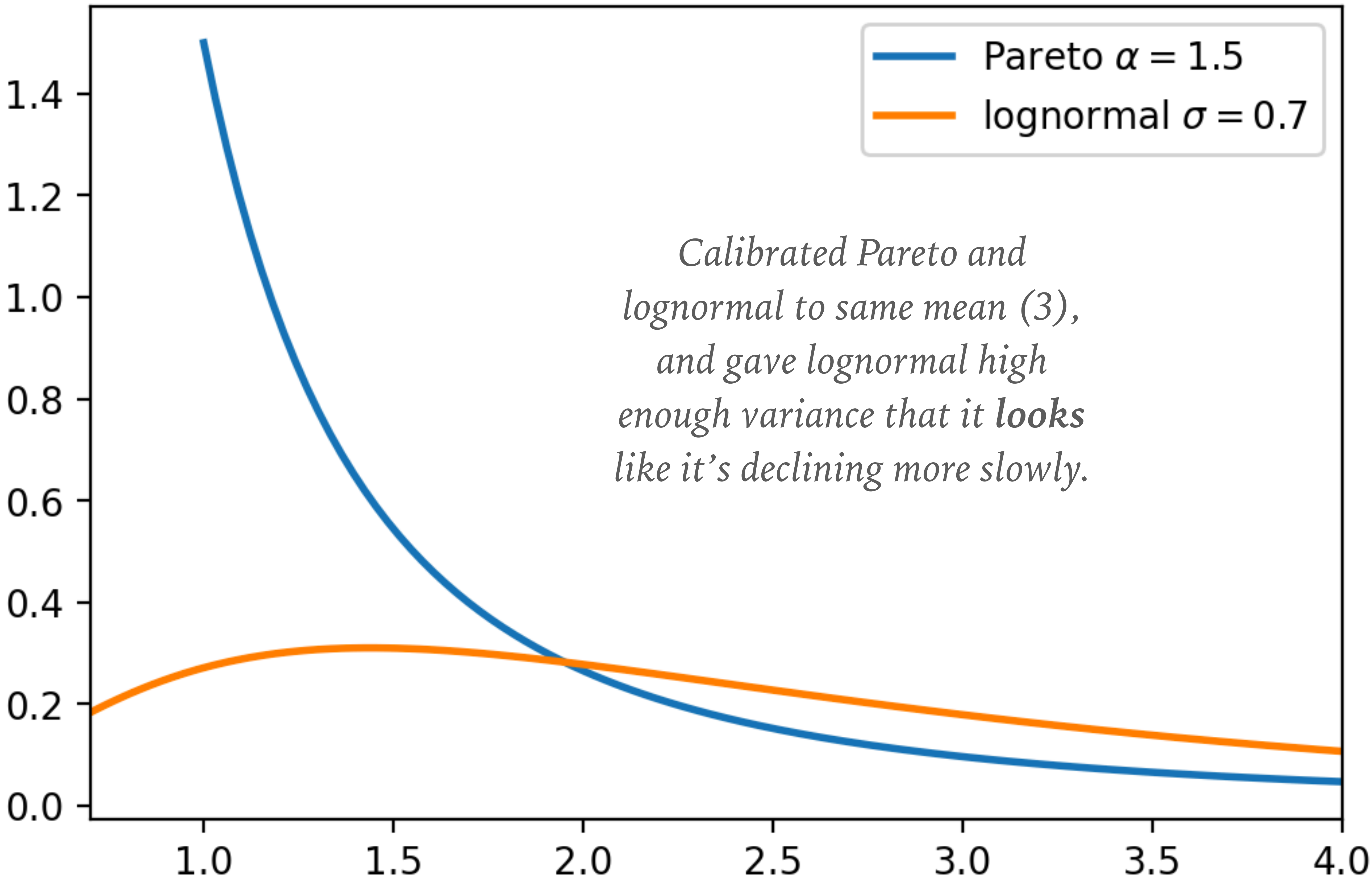


NICE BASIC PROPERTIES OF THE PARETO DISTRIBUTION

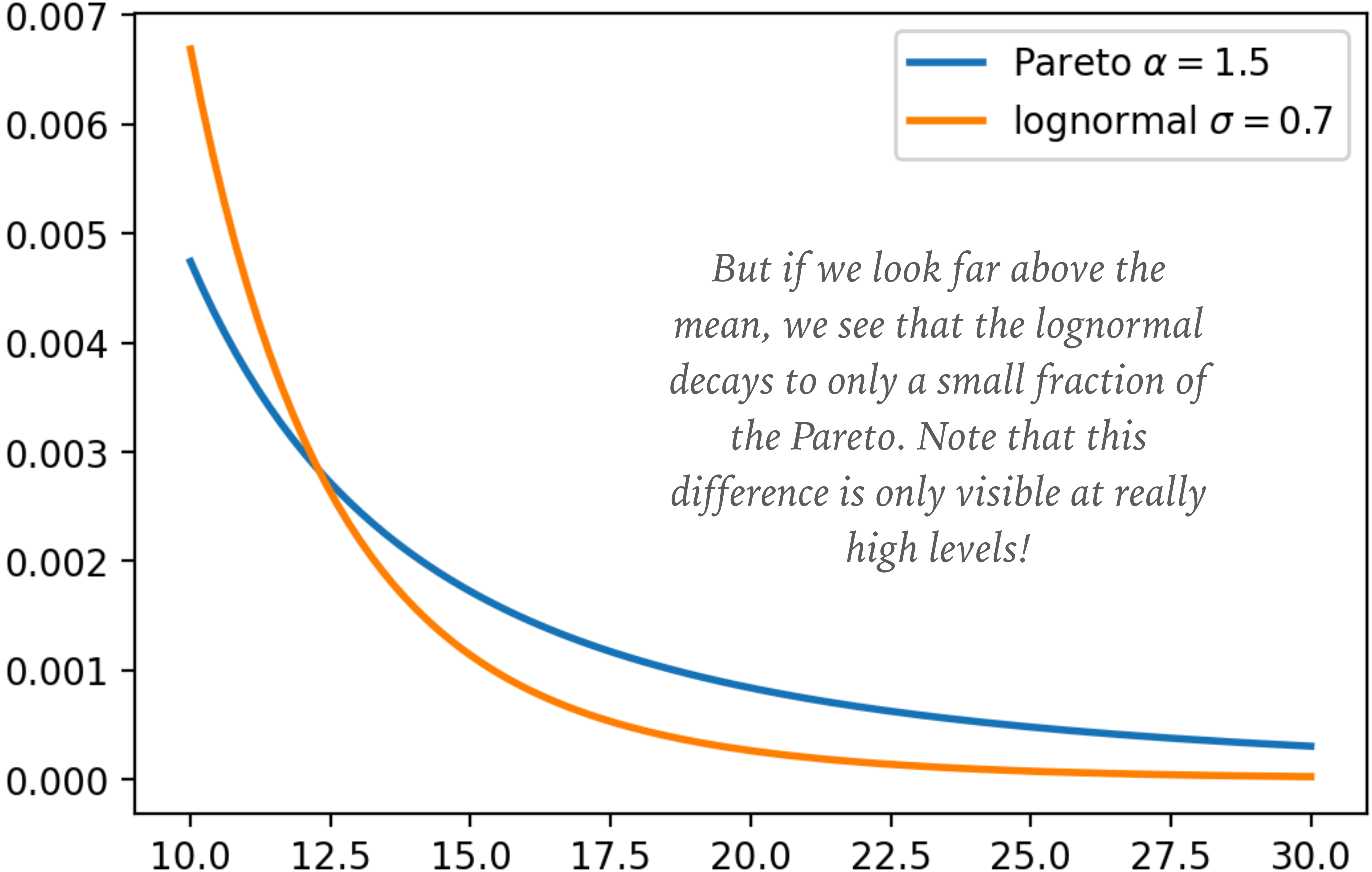
- If you cut it off at some higher x_m, it's **still Pareto** with the same shape parameter α
- The **mean**, assuming $\alpha > 1$, is given by $\frac{\alpha}{\alpha - 1} x_m$
  - so if you ask "what's the average wealth among people who hold at least x_m", the answer is $\alpha/(\alpha - 1)$ times x_m
  - first moment doesn't exist for $\alpha \leq 1$
  - in general, only moments greater than α exist

- The **log** of a Pareto is **exponentially distributed**

COMPARING VS. LOGNORMAL: DENSITIES

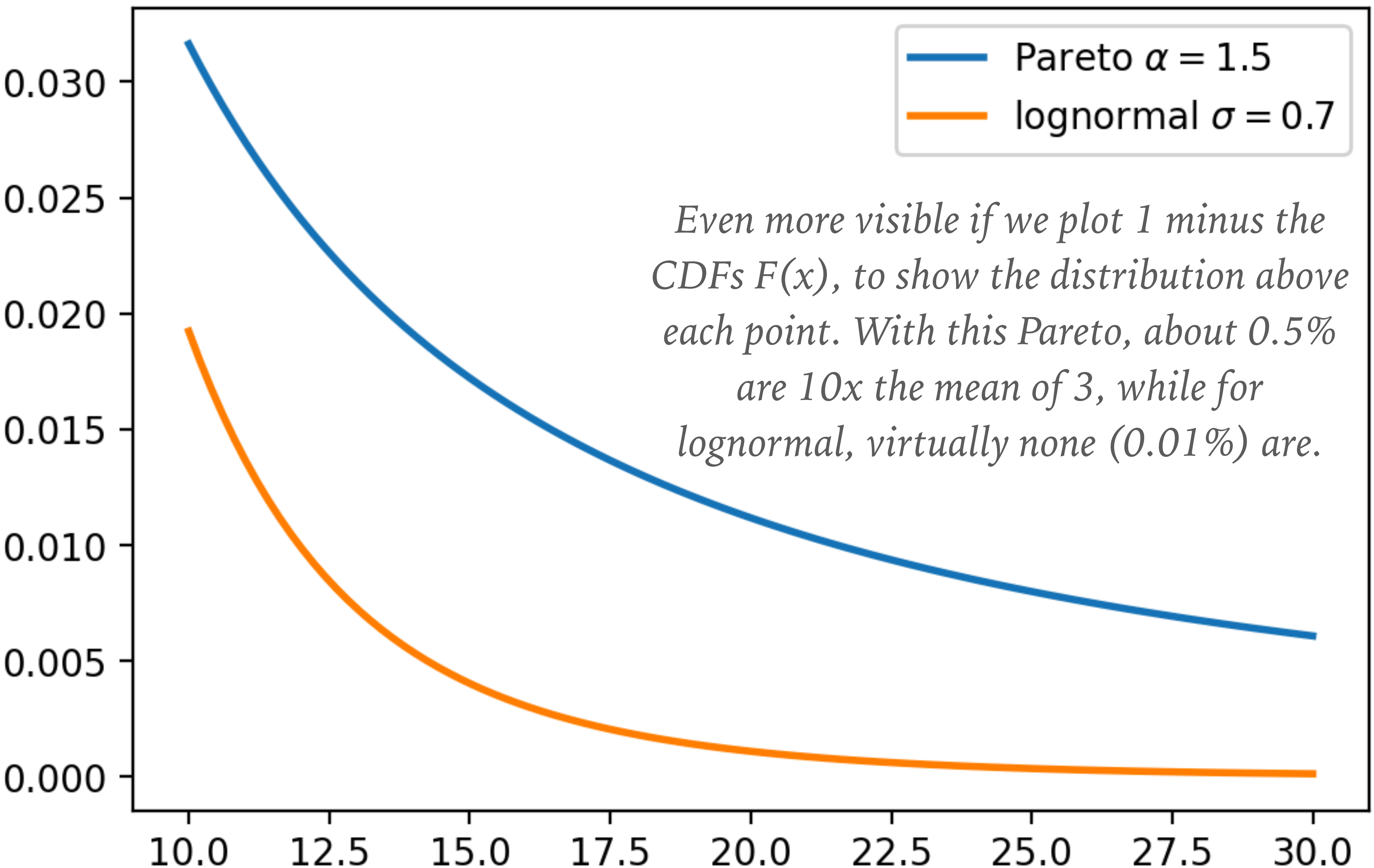


COMPARING VS. LOGNORMAL: DENSITIES



But if we look far above the mean, we see that the lognormal decays to only a small fraction of the Pareto. Note that this difference is only visible at really high levels!

COMPARING VS. LOGNORMAL: COMPLEMENTARY CDF 1−F(X)



ANOTHER NICE FEATURE OF PARETO: "DENSITY OF DOLLARS"

- The density of a Pareto is uniquely defined by the fact that it starts at x_m and is proportional to $f(x) \propto x^{-\alpha-1}$
- What if we look at density of **dollars** rather than of **people**?
- Density of dollars held by people with wealth x is proportional to $xf(x) \propto x^{-\alpha}$, Pareto with shape $\alpha - 1$!

- So:
  - if wealth of **people** is Pareto with shape α, then
  - distribution of "how rich are the people who hold each **dollar** of wealth" is Pareto with shape $\alpha - 1$

Fatter tail because dollars are more likely to be held by wealthier people!

HOW CAN WE USE THIS FACT?

- Threshold x^* for top c percent is given by

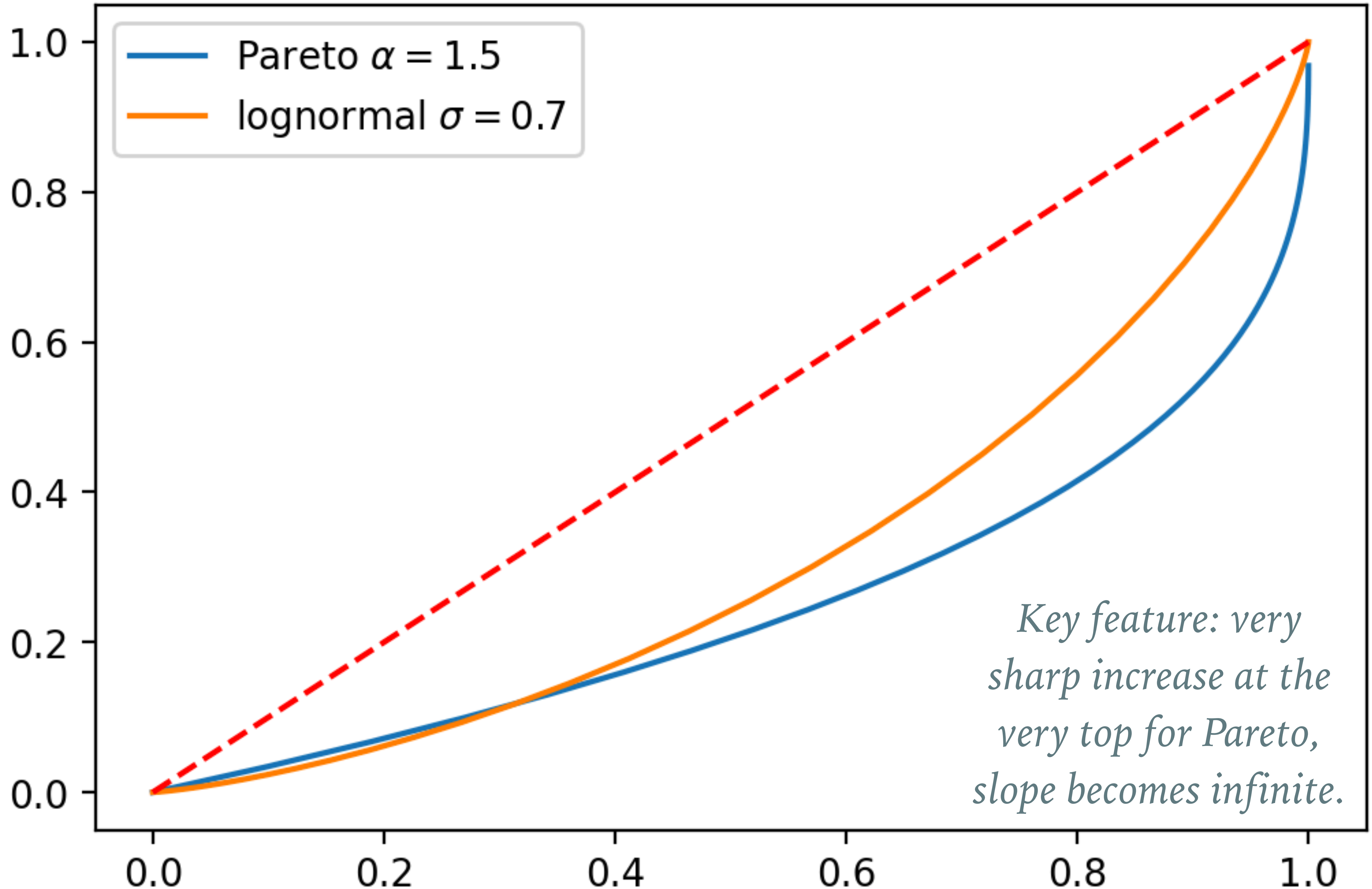
$$1 - F(x^*) = c \rightarrow \left(\frac{x_m}{x^*}\right)^{\alpha} = c \rightarrow x^* = c^{-1/\alpha} x_m$$

- If you want to ask "what share of dollars are held by the top c percent", use distribution of dollars, which has shape $\alpha - 1$

$$1 - F^{dol}(x^*) = \left(\frac{x_m}{x^*}\right)^{\alpha-1} = c^{\frac{\alpha-1}{\alpha}}$$

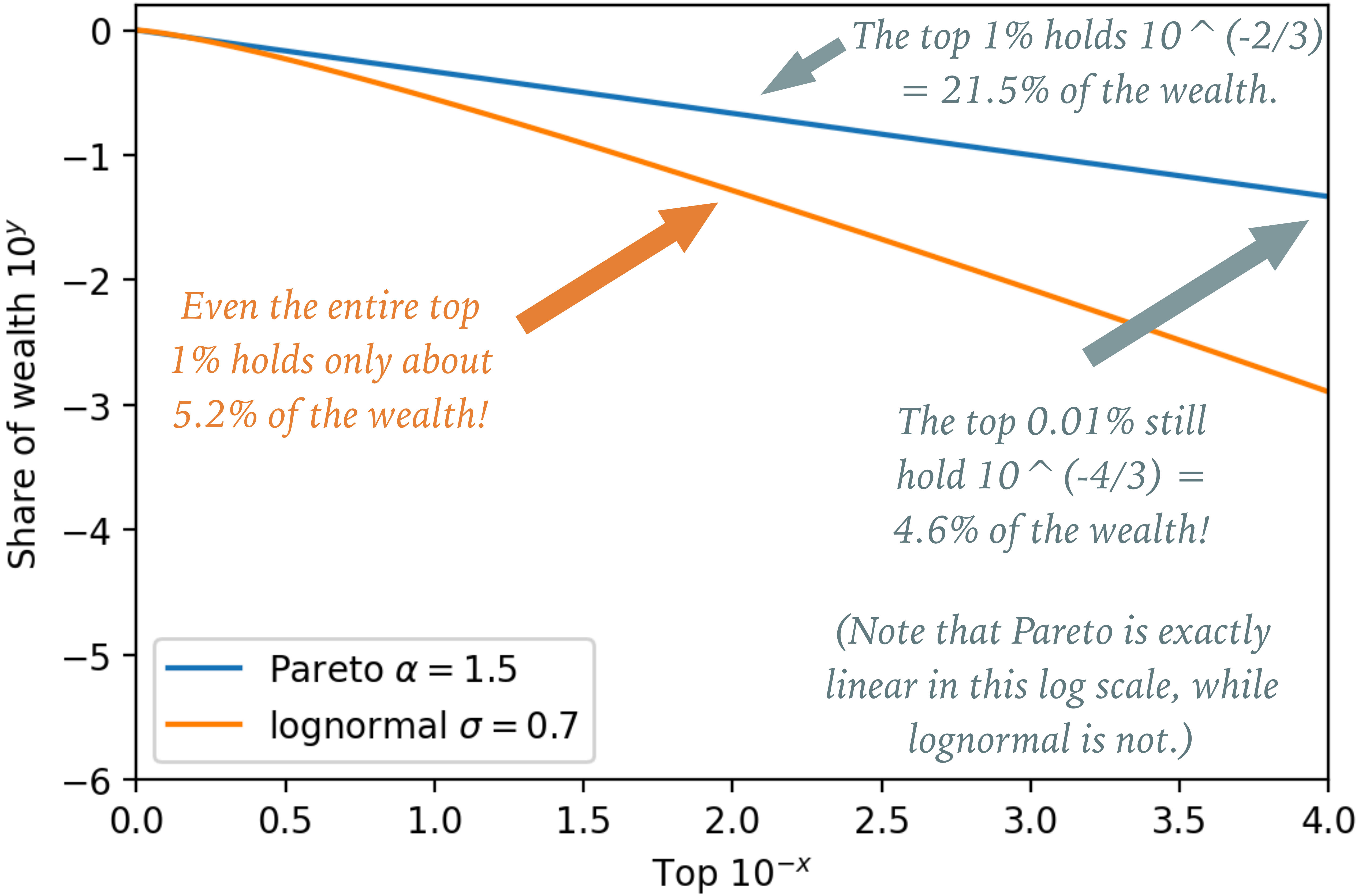
- So if $\alpha = 1.5$, $c^{\frac{\alpha-1}{\alpha}} = 0.1^{1/3} = 46\,\%$ held by top $10\,\%$
- Caution: usually Pareto only describes the tail, so absolute shares from this aren't right. But $c^{\frac{\alpha-1}{\alpha}}$ still gives *relative* shares!

LORENZ CURVES (CDF OF DOLLARS VS. CDF OF PEOPLE)



Key feature: very sharp increase at the very top for Pareto, slope becomes infinite.

PLOT 1 MINUS THESE CDFS ON LOGARITHMIC SCALE



PARETO TAILS OF INCOME AND WEALTH

PARETO TAILS ARE EVERYWHERE

- Not many variables have exact Pareto for entire distribution
  - (sharp minimum x_m too unrealistic)
- But lots have Pareto **tail**: if we cut off at high x_m, it's Pareto
- Zipf's law, the special case $\alpha = 1$, famously holds for things like word frequencies and city sizes, over a wide range
- We will be interested in Pareto tails for **income** and **wealth**

WEALTH INEQUALITY: APPROXIMATE PARETO TAIL

- Saez and Zucman 2019 update (note wealth inequality is controversial, and they come in on higher end):
  - Top 10%: 77.3%
  - Top 1%: 37.9%
  - Top 0.1%: 19.2%

Extrapolating with 1.42, top 0.00001% holds 0.85% of the wealth, close to Saez-Zucman estimate of just above 1%

- Recall: in Pareto tail, share held by top c is prop to $c^{\frac{\alpha-1}{\alpha}}$
- Back out $\frac{\alpha-1}{\alpha}$ from relative observations, pretty close:

$$\approx \frac{\log(77.3) - \log(37.9)}{\log(10)} \approx 0.31 \qquad \approx \frac{\log(37.9) - \log(19.2)}{\log(10)} \approx 0.295$$

- Both imply α between 1.42 and 1.45

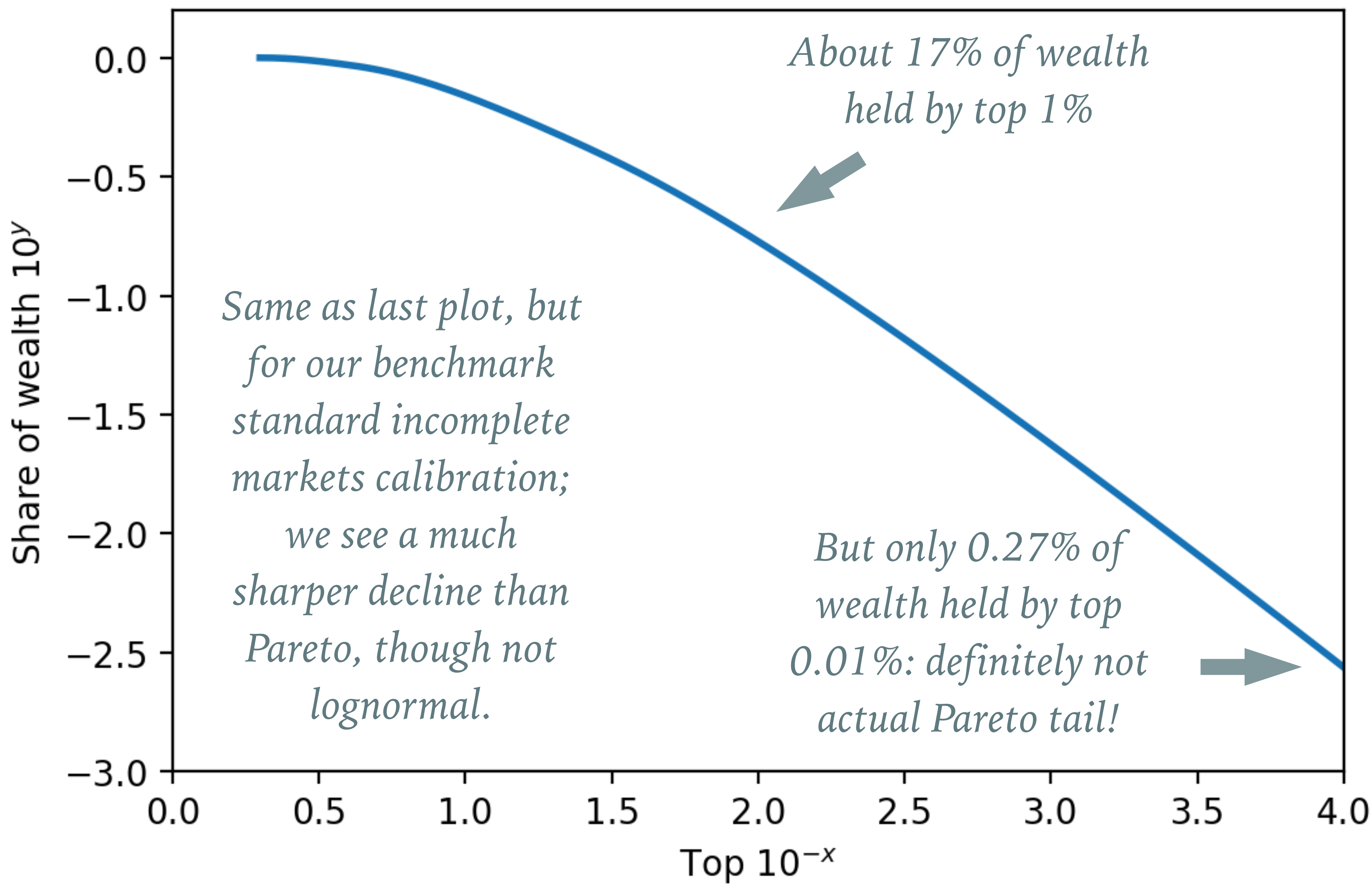
INCOME INEQUALITY: APPROXIMATE PARETO TAIL

- Piketty-Saez (2019 update, excluding capital gains):
  - Top 10%: 47.12%
  - Top 1%: 17.59%
  - Top 0.1%: 7.21%
  - Top 0.01%: 2.92%
- Back out $\frac{\alpha - 1}{\alpha}$ from relative observations, pretty close:

$$\frac{\log(47.12) - \log(17.59)}{\log(10)} \approx 0.428 \qquad \frac{\log(17.59) - \log(7.21)}{\log(10)} \approx 0.387 \qquad \frac{\log(7.21) - \log(2.92)}{\log(10)} \approx 0.393$$

- Second two (more relevant for tail) imply α of about 1.64
  - fat tail, but **thinner than wealth!**

WHAT DO THE CALIBRATIONS WE'VE USED IMPLY FOR TAIL WEALTH?



BUT WE HAD LOGNORMAL INCOME, WHAT IF WE HAD PARETO?

- Benhabib, Bisin, Luo (2017) and others: if the income distribution has a Pareto tail, the wealth distribution in the standard incomplete markets model has a Pareto tail **with the same Pareto shape parameter**
  - we had $\alpha = 1.42$ for wealth and $\alpha = 1.64$ for income
  - so we can't match tail wealth inequality even if we recalibrate model to match Pareto for tail income (vs. current lognormal)
  - why this result? asymptotic asset policy function in the model is linear with slope a bit below 1, really high wealth is just driven by getting high incomes a bunch of times, no mechanism for asymptotically higher wealth dispersion
- **How can we fix this?**

ANSWER: FATTER PARETO TAILS FOR WEALTH ABOUT GROWTH RISK

- Need there to be risk that affects wealth **multiplicatively**
- One example: move to continuous time and suppose

$$da_t = [w + (\bar{r} - \bar{c})a_t]dt + \sigma a_t dW_t$$

- Here, $\bar{r}$ is mean return on wealth, $\bar{c}$ is (we'll take exogenous) consumption rate out of wealth, w is exogenous wage income, and $\sigma a_t dW_t$ is multiplicative risk to wealth with volatility σ
- Then: wealth distribution has Pareto tail with parameter

$$\alpha = 1 + \frac{\bar{c} - \bar{r}}{\sigma^2/2} > 1$$

(cf Moll's notes on Piketty, https://benjaminmoll.com/wp-content/uploads/2019/07/piketty_notes.pdf)

ANALYZING THE FORMULA

- If we assume income w grows at rate g, then wealth distribution detrended by g has tail

$$\alpha = 1 + \frac{\bar{c} + g - \bar{r}}{\sigma^2/2} > 1$$

- One basis of Thomas Piketty talking about $\bar{r} - g$ and wealth inequality, since fatter tail when $\bar{r} - g$ larger
- Also fatter tail when shocks σ larger

A SIMPLER MODEL: PART 1

- Still continuous time
- Assume at date t, new people are born at rate e^{nt}
  - n is rate of growth of newborn population
- Death occurs, and any wealth dissipates, at constant rate μ
- So, at any t, the age-j cohort has population size $e^{n(t-j)-\mu j}$
  - So, within the population at t, the distribution of ages is exponential, with CDF $F(j) = 1 - e^{-(n+\mu)j}$

- Can think of this model loosely as characterizing large intergenerational accumulations of wealth, not just literal lives

A SIMPLER MODEL: PART 2

- Assume new people born at t start with wealth $e^{\gamma t}$
- Wealth earns return $\bar{r}$, people consume from it at rate $\bar{c}$
- So, wealth of age-j people at time t is $a_{jt} = e^{\gamma(t-j)+(\bar{r}-\bar{c})j}$
  - We'll assume that $\bar{r} > \bar{c} + \gamma$, so older are richer
- Fraction of population older than j is $e^{-(n+\mu)j}$
- So, if G_t is CDF of *wealth* at date t, we have:

$$1 - G_t(a_{jt}) = e^{-(n+\mu)j} \qquad j = \frac{\log a_{jt} - \gamma t}{\bar{r} - \bar{c} - \gamma}$$

$$1 - G_t(a_{jt}) \propto a_{jt}^{-\frac{n+\mu}{\bar{r}-\bar{c}-\gamma}}$$

A SIMPLER MODEL: CONCLUSION

- Any asset level $a \geq a_{0t}$ corresponds to some age j, so last slide gave us a formula for distribution G_t:

$$1 - G_t(a) \propto a^{-\frac{n+\mu}{\bar{r} - \bar{c} - \gamma}}$$

- This is **Pareto** with shape parameter $\alpha = \dfrac{n+\mu}{\bar{r} - \bar{c} - \gamma}$
  - In general, we get Pareto from *exponential growth over exponentially distributed time,* like we have here
- α is falling in $\bar{r}$: higher returns increase thickness of wealth tail
- α is rising in $\bar{c}$, γ, n, and μ: higher consumption rates, faster growth, and more dissipation of wealth decrease thickness of wealth tail