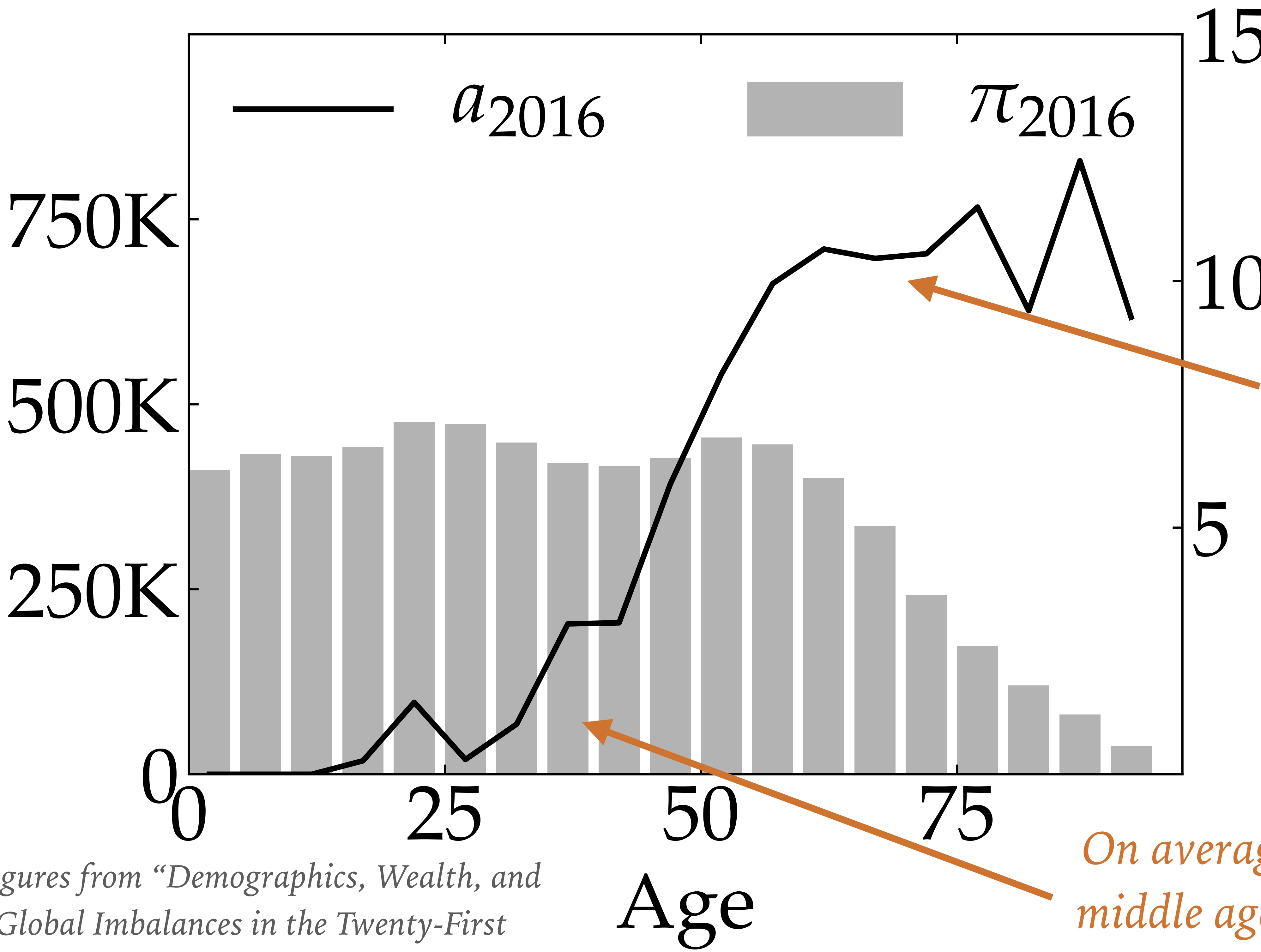
LIFE-CYCLE / OVERLAPPING GENERATIONS MODELS

Econ 411-3
Matthew Rognlie, Spring 2024

AVERAGE ASSET-HOLDINGS EXTREMELY CORRELATED WITH AGE!

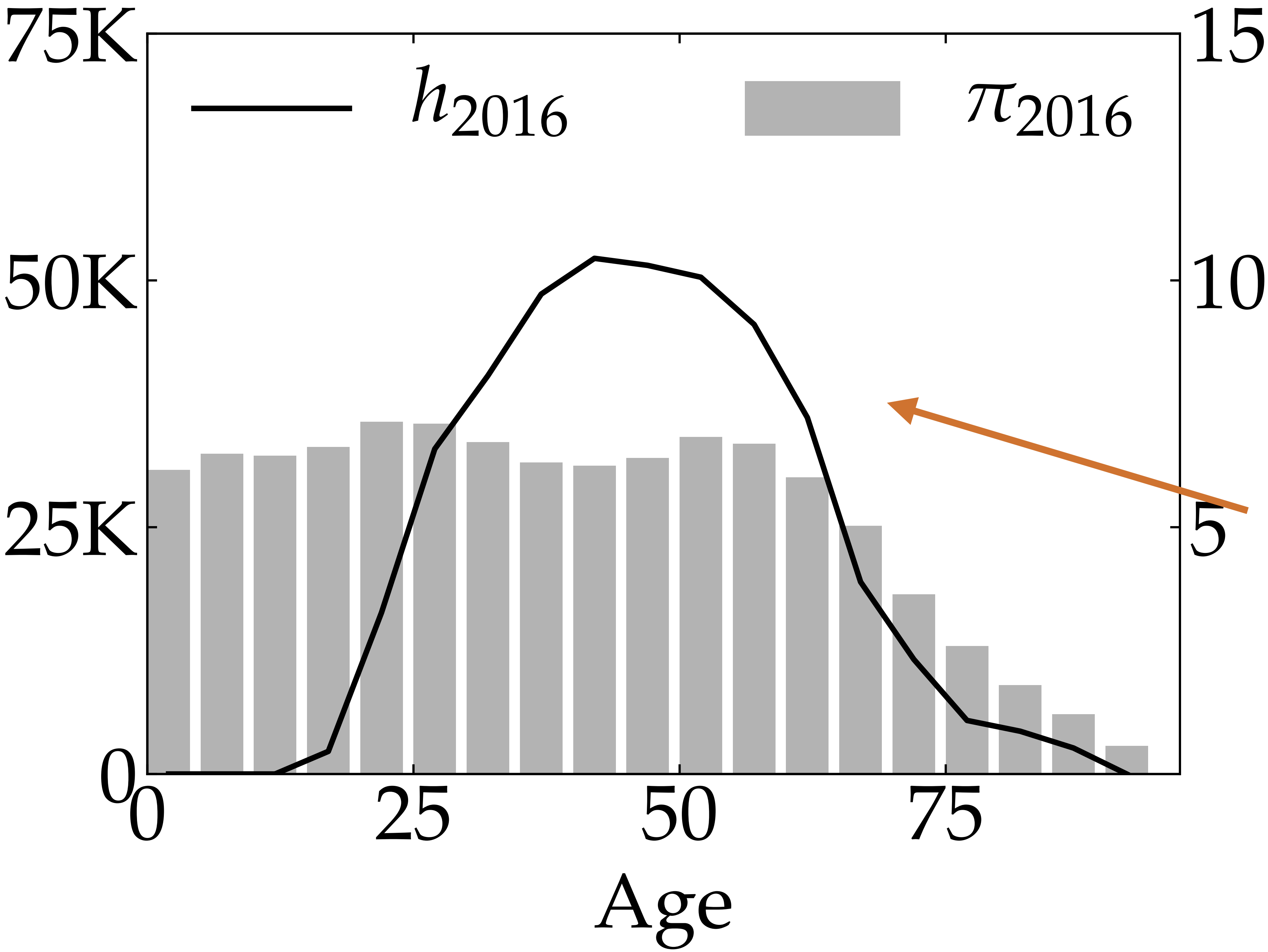


Interestingly, average wealth reaches a plateau, rather than being drawn down, around retirement age

On average, people prior to middle age hold almost zero wealth.

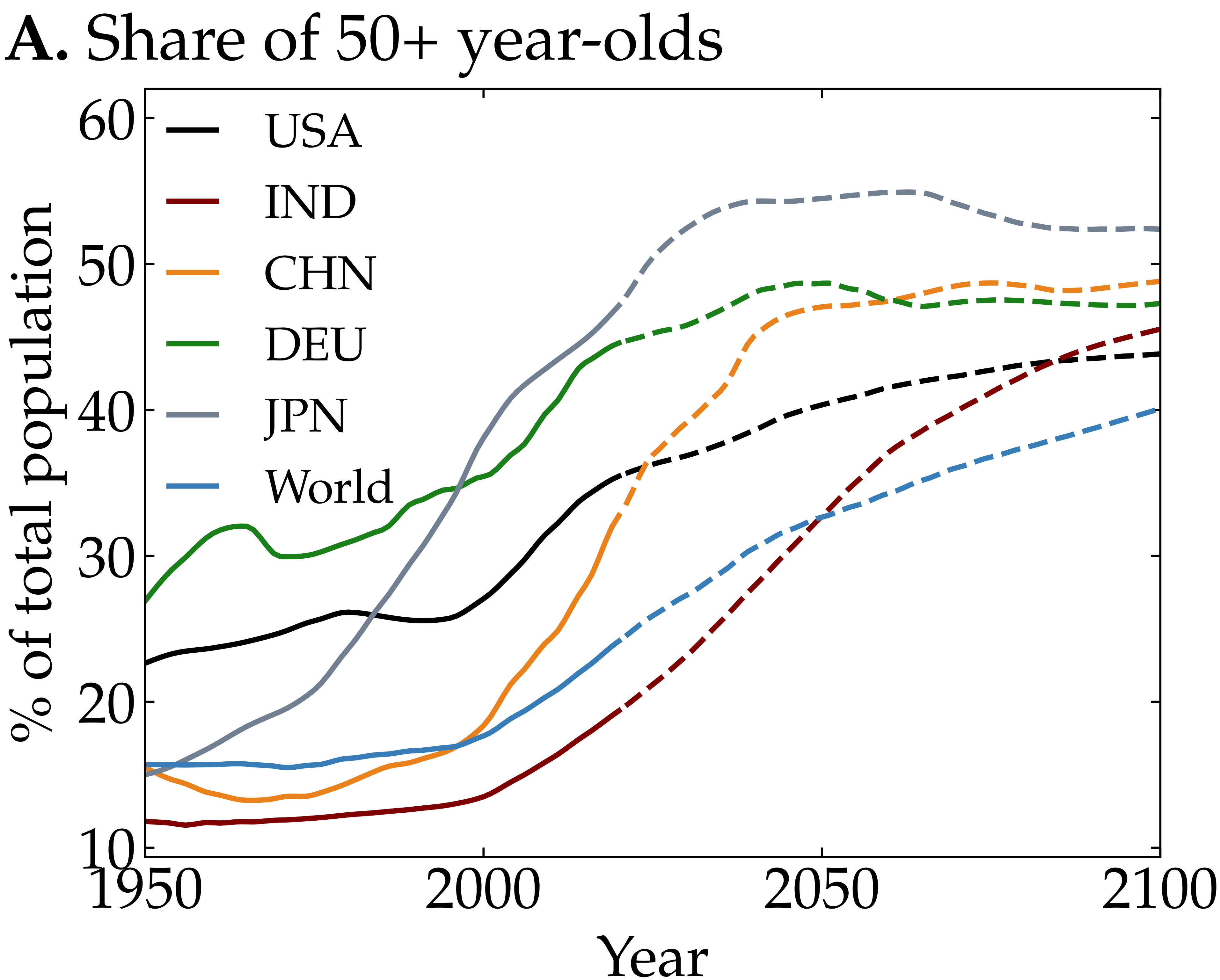
(Figures from "Demographics, Wealth, and Global Imbalances in the Twenty-First Century";this is for US, but similar elsewhere)

DISTINCT BUT ALSO VERY STRONG PATTERN IN LABOR INCOME



Average labor income is bell-shaped, rising to reach a sustained mid-life peak, then falling rapidly as retirement kicks in.

GLOBAL POPULATION IS RAPIDLY AGING



This is due to a combination of increasing longevity and falling fertility, with fertility likely to be biggest contributor. (This is from 2019 UN projections; ***updated figures likely more extreme!****)*

ONLY GETTING MORE EXTREME WITH TIME…

Asia Pacific

In South Korea, world's lowest fertility rate plunges again in 2023

By **Jihoon Lee** and **Cynthia Kim**

February 28, 2024 8:16 AM CST · Updated a month ago

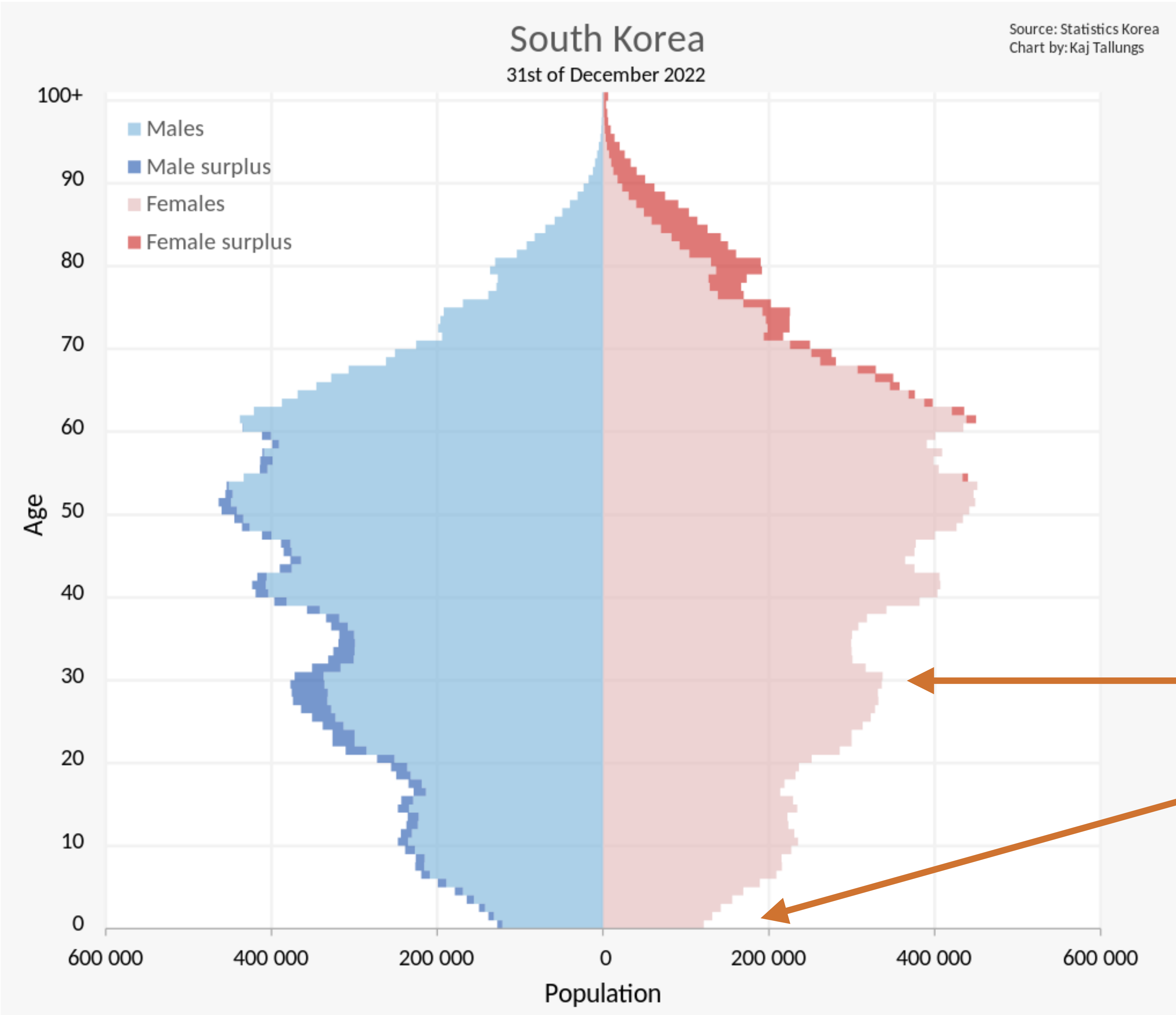
Aa

SEOUL, Feb 28 (Reuters) - South Korea's fertility rate, already the world's lowest, continued its dramatic decline in 2023, as women concerned about their career advancement and the financial cost of raising children decided to delay childbirth or to not have babies.

The average number of expected babies for a South Korean woman during her reproductive life fell to a record low of 0.72 from 0.78 in 2022, data from Statistics Korea showed on Wednesday.

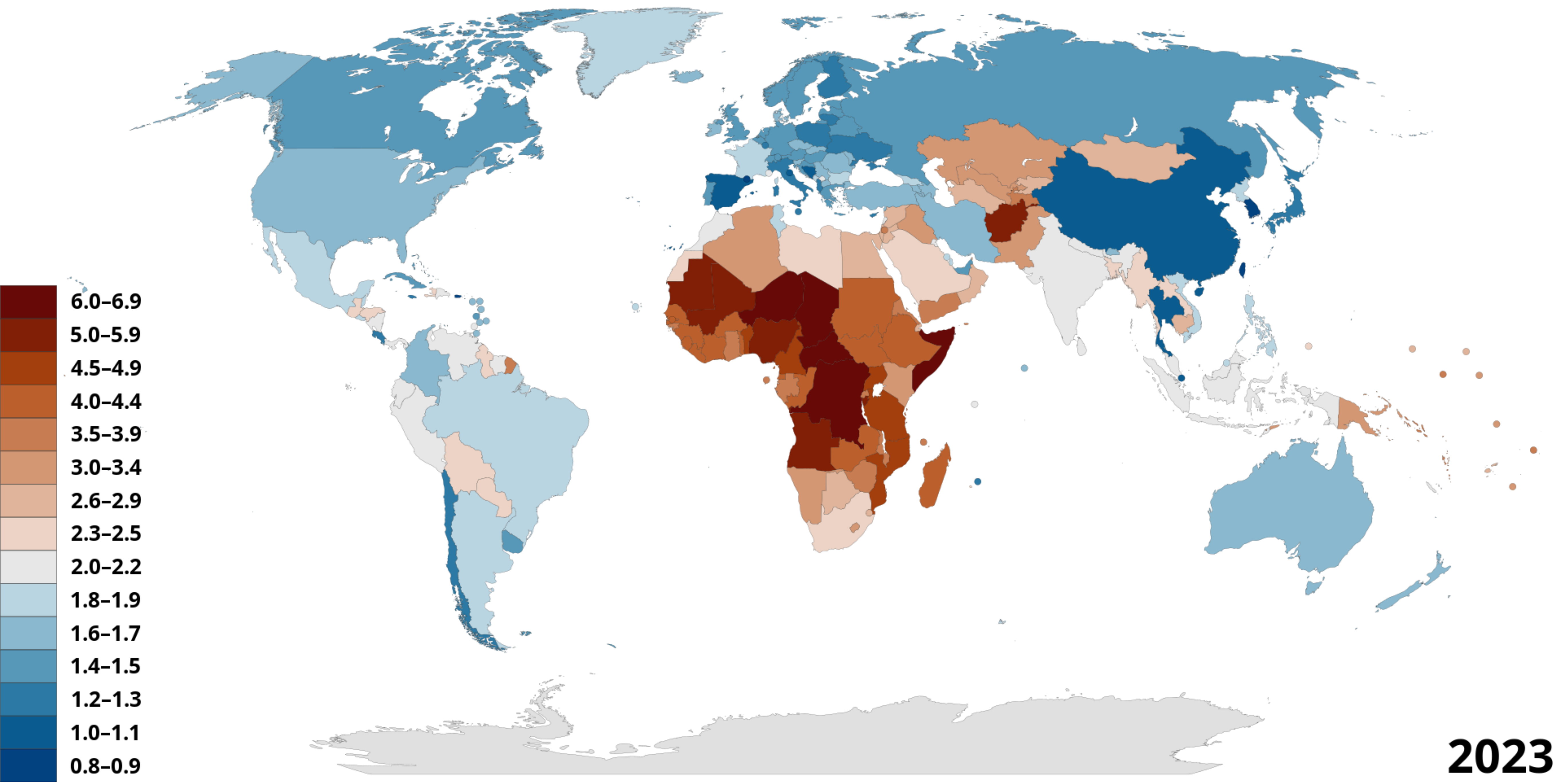
*(A total fertility rate of **0.72** is about 1/3 of replacement rate; each successive generation will be 1/3 the size of its predecessor.)*

IMPLIES A MASSIVE SHIFT IN RELATIVE AGE STRUCTURE



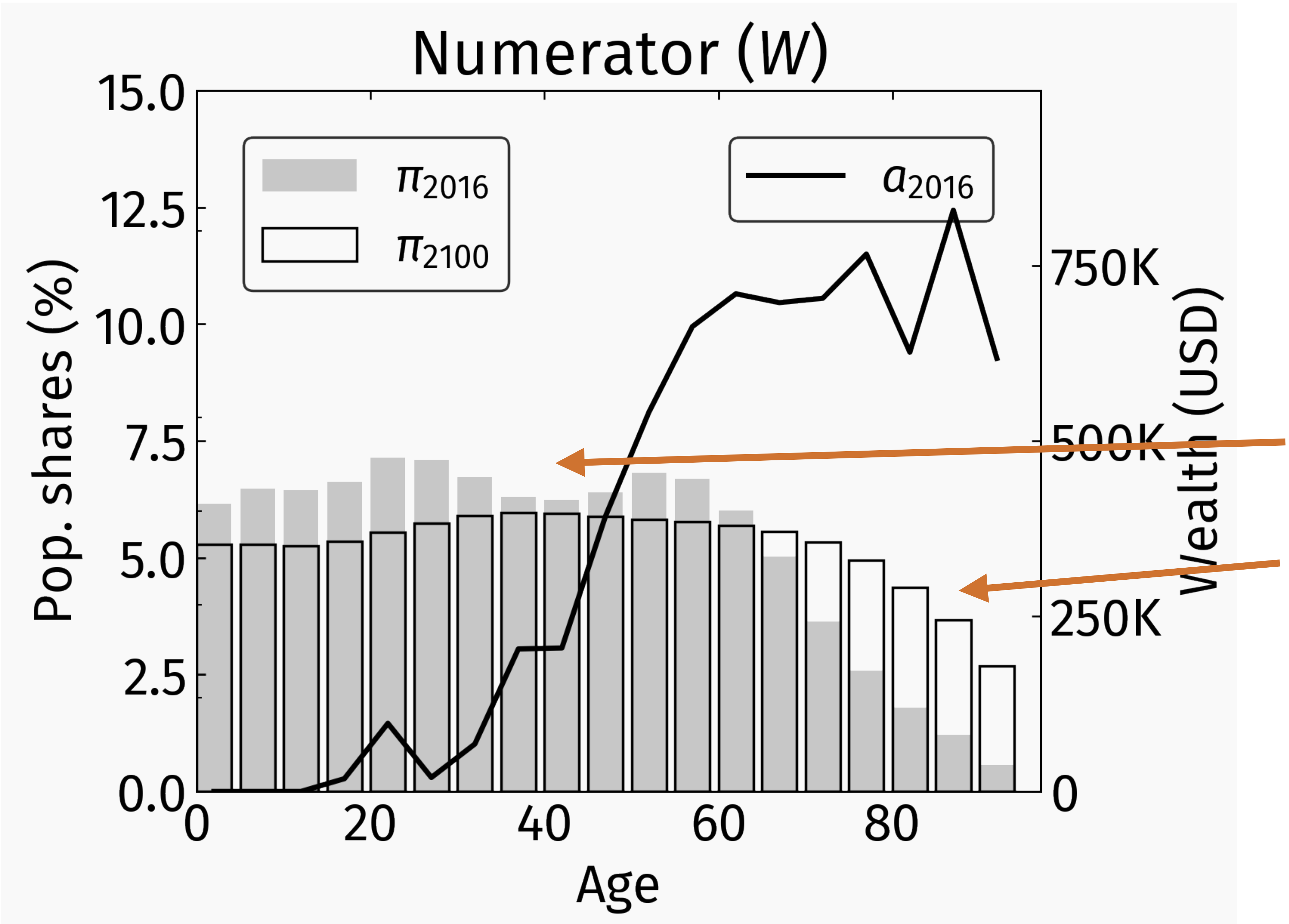
Only starting to become visible, but if today's babies are 1/3 as numerous as the 30-year-olds, then there will be ~1/3 as many 40-year olds to provide for 70-year olds)

HAPPENING BROADLY ACROSS MOST PARTS OF THE WORLD



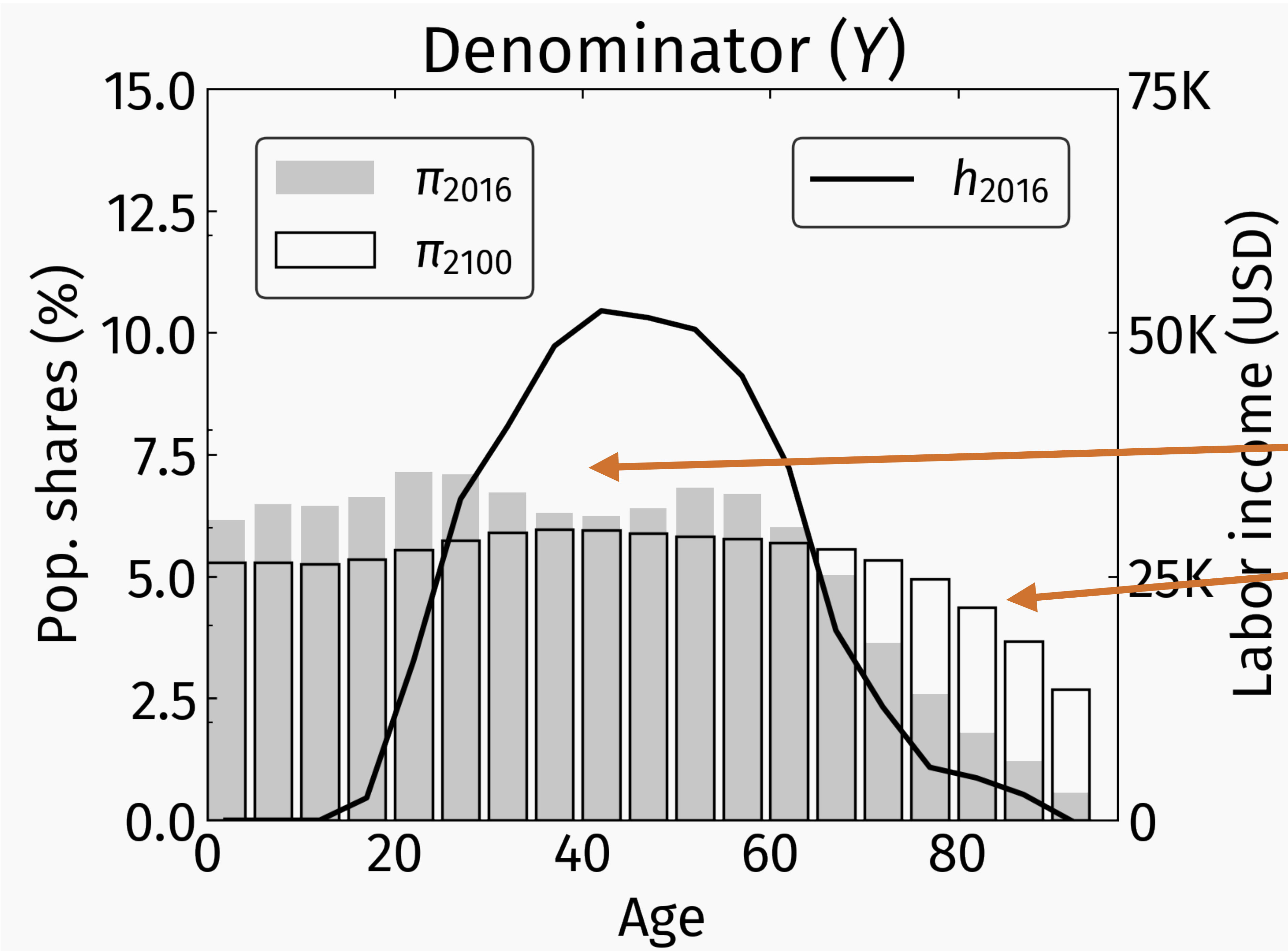
Blue colors mean below-replacement fertility, where (absent immigration) older generations will outnumber younger ones.

A MAJOR INCREASE IN ASSET DEMAND…



Big projected shift in population distribution to older ages, who hold lots more assets.

AND A MAJOR DECREASE IN LABOR SUPPLY?



Shift toward older ages who earn very little labor income, suggesting a decline in effective per capita labor supply (unless retirement happens much later!)

TWO BIG REASONS TO STUDY MODELS WITH EXPLICIT LIFE CYCLE

- First, it's a crucial aspect of real-life consumption-saving behavior
  - one of the main motivations to save is to build assets for retirement (and sometimes bequests)
  - age is one conspicuous dimension of heterogeneity in asset holdings
- Second, the shift to a much older population is itself a huge shock, maybe the biggest macro shock we'll expect to face
  - huge increase in asset demand, decrease in effective labor supply

A CANONICAL LIFE-CYCLE MODEL

(Simplified version of model in section 2 of "Demographics, Wealth, and Global Imbalances in the Twenty-First Century", with idiosyncratic risk removed.)

A CANONICAL LIFE-CYCLE MODEL

- Age $j = 0$ is beginning of (adult) life, $j = J$ maximum possible

- Survive from j to $j + 1$ with probability ϕ_j, define cumulative survival probability $\Phi_j \equiv \prod_{k=0}^{j-1} \phi_k$ up to age j, with $\Phi_0 = 1$

- Earn exogenous income y_j at age j, no other risk

- Start life with $a_0 = 0$, choose paths of $\{c_j, a_{j+1}\}$

A CANONICAL LIFE-CYCLE MODEL: FULL OPTIMIZATION PROBLEM

$$\max_{\{c_j, a_j\}} \sum_{j=0}^{J} \beta_j \Phi_j \frac{c_j^{1-\frac{1}{\sigma}} - 1}{1 - \frac{1}{\sigma}}$$

$$s.t.\ c_j + \phi_j a_{j+1} = y_j + (1+r)a_j$$

$$a_0 = 0$$

- CRRA preferences with EIS σ
- Utility at age j scaled by survival probability, Φ_j, and shifter β_j for generality (geometric discounting $\beta_j \equiv \beta^j$ special case)
- Key feature: to save a_{j+1} for next period, only need to spend $\phi_j a_{j+1}$ today, scaled by probability ϕ_j of survival
  - i.e. all saving is done via actuarially fair "annuity" accounts

INTERTEMPORAL EULER EQUATION

$$\max_{\{c_j, a_j\}} \sum_{j=0}^{J} \beta_j \Phi_j \frac{c_j^{1-\frac{1}{\sigma}} - 1}{1 - \frac{1}{\sigma}}$$

$$s.t.\ c_j + \phi_j a_{j+1} = y_j + (1+r) a_j$$

$$\beta_j \Phi_j c_j^{-1/\sigma} = \frac{1+r}{\phi_j} \beta_{j+1} \Phi_{j+1} c_{j+1}^{-1/\sigma}$$

(Euler equation between consumption at j and j+1)

Survival odds cancel out since $\Phi_{j+1} = \phi_j \Phi_j$!

$$\beta_j c_j^{-1/\sigma} = (1+r) \beta_{j+1} c_{j+1}^{-1/\sigma}$$

$$\beta_j c_j^{-1/\sigma} = (1+r)^{k-j} \beta_k c_k^{-1/\sigma}$$

(Euler equation between consumption at any j and k)

PRESENT-VALUE INTERTEMPORAL BUDGET CONSTRAINT

$$c_j + \phi_j a_{j+1} = y_j + (1+r)a_j$$

Multiply by $(1+r)^{-j}\Phi_j$ and rearrange

$$(1+r)^{-j}\Phi_j(y_j - c_j) = (1+r)^{-j}\Phi_{j+1}a_{j+1} - (1+r)^{-j+1}\Phi_j a_j$$

Sum across all $j = 0, \ldots, J$

$$\sum_{j=0}^{J}(1+r)^{-j}\Phi_j(y_j - c_j) = \sum_{j=0}^{J}(1+r)^{-j}\Phi_{j+1}a_{j+1} - \sum_{j=0}^{J}(1+r)^{-j+1}\Phi_j a_j$$

Given $a_0 = 0$ and $\Phi_{J+1} = 0$, these two sums have the same nonzero terms, and cancel out

$$\sum_{j=0}^{J}(1+r)^{-j}\Phi_j(y_j - c_j) = 0$$

A simple survival-weighted present-value budget constraint!

SUMMARY OF RESULTS SO FAR

- Intertemporal Euler equation between any pair of ages:

$$\beta_j c_j^{-1/\sigma} = (1+r)^{k-j} \beta_k c_k^{-1/\sigma}$$

- (Survival-weighted) present-value budget constraint:

$$\sum_{j=0}^{J} (1+r)^{-j} \Phi_j (y_j - c_j) = 0$$

- **Together these characterize the solution to the model!**
- Ultimately **very simple**: individuals effectively have complete markets with respect to longevity, and can choose what fraction of expected lifetime resources to allocate to each age

COMPUTATION

- Write consumption at each j in terms of age-0 consumption:

$$c_j = \underbrace{(1+r)^{\sigma j}\left(\frac{\beta_j}{\beta_0}\right)^{\sigma}}_{\equiv \theta_j} c_0$$

- Plug into budget constraint and solve for c_0:

$$c_0 = \frac{\sum_{j=0}^{J}(1+r)^{-j}\Phi_j y_j}{\sum_{j=0}^{J}(1+r)^{-j}\Phi_j \theta_j}$$

- (Then use this c_0 to calculate all $c_j = \theta_j c_0$.)

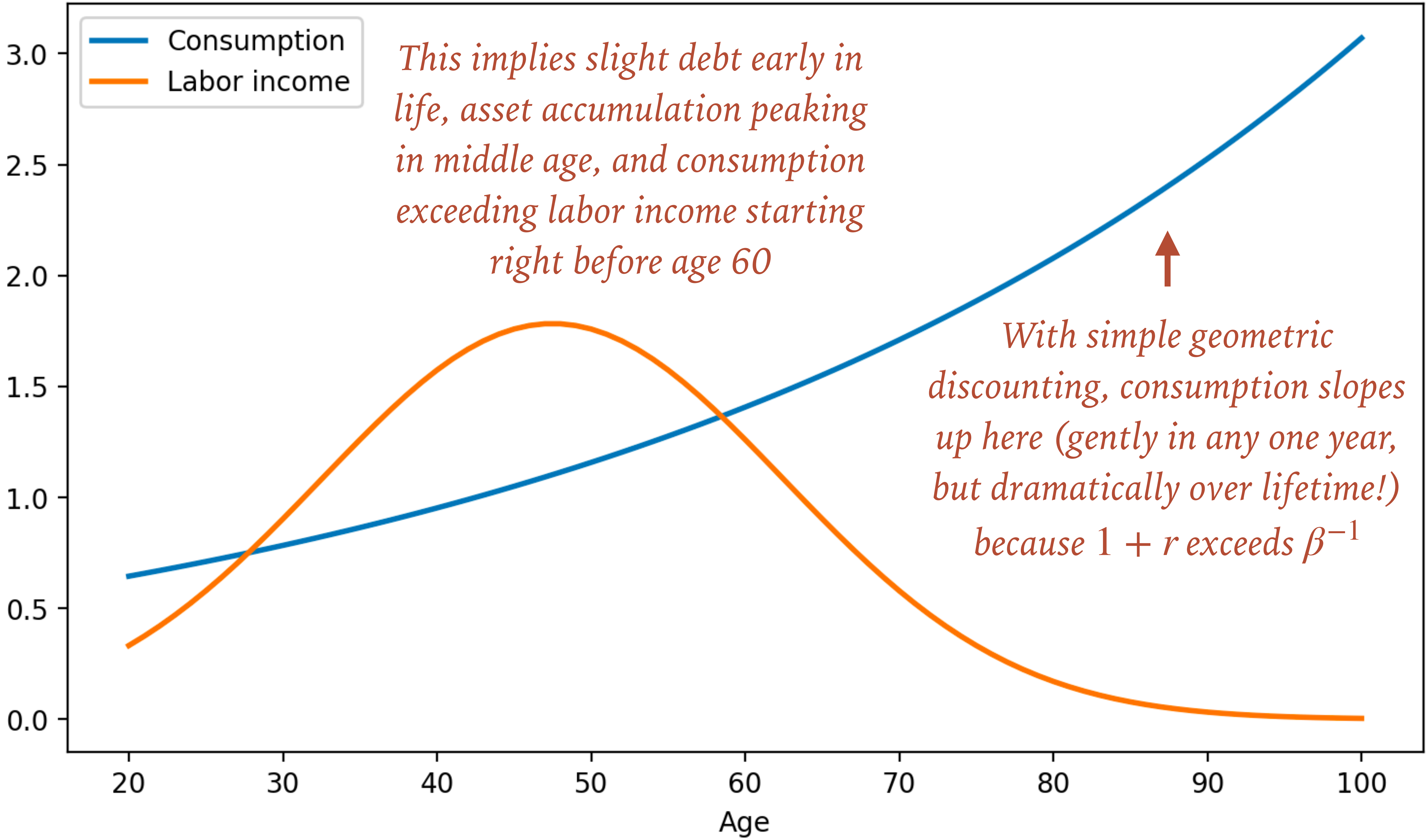
COMPUTATION, CONTINUED

- Once we have c_j at all ages from last slide, we can iterate on budget constraint to obtain a_j at all ages
- Numerically more stable to iterate backward from end of life, starting with initial condition $\phi_J a_{J+1} = 0$:

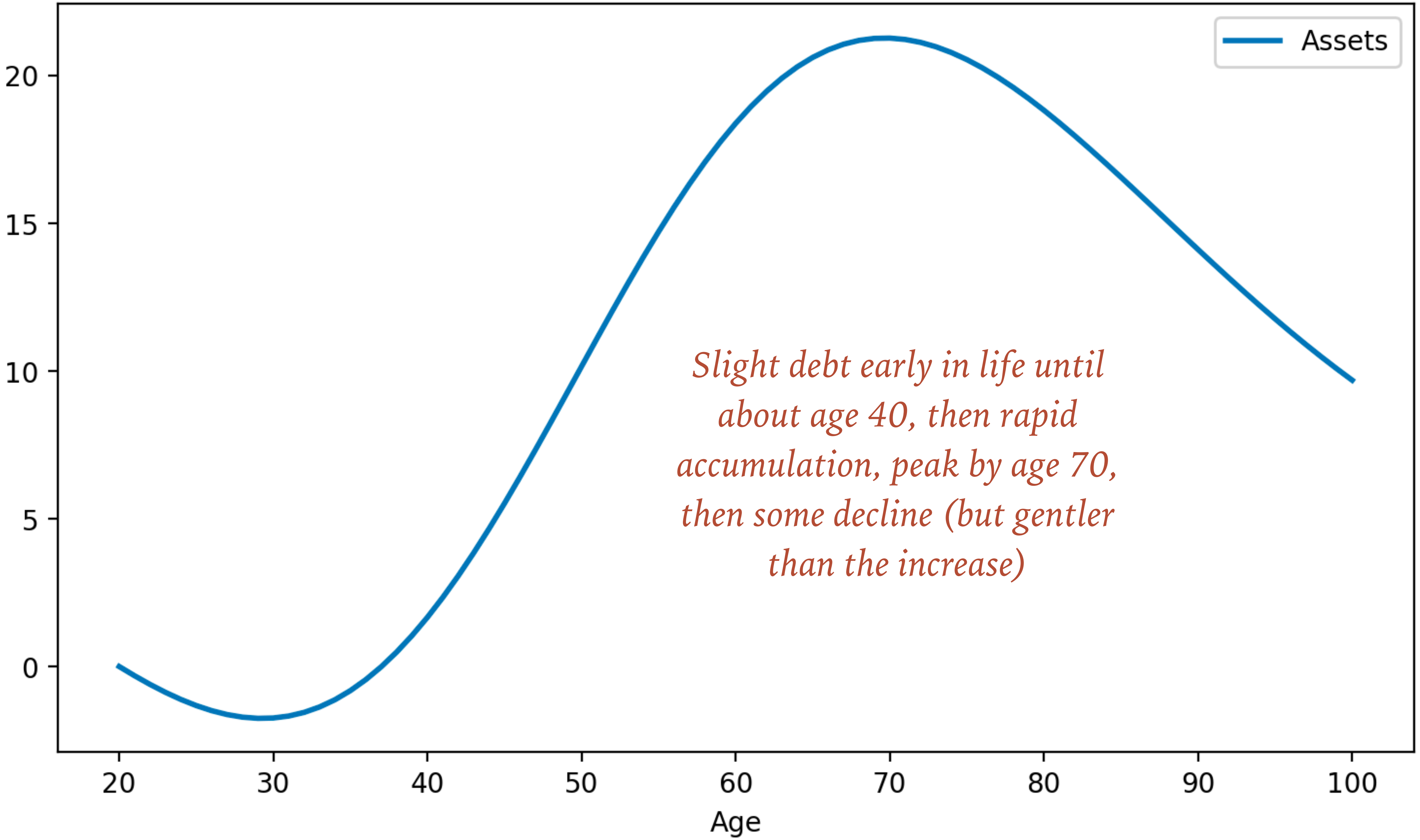
$$a_j = \frac{c_j - y_j + \phi_j a_{j+1}}{1+r}$$

- We'll now try a simple application using empirically realistic "Gompertz" survival function for Φ_j and a roughly reasonable bell shape for income y_j, plus $\sigma = 1$, $r = 0.03$ and $\beta_j = 0.99^j$, with $j = 0,\ldots,99$, and $j = 0$ corresponding to age 20

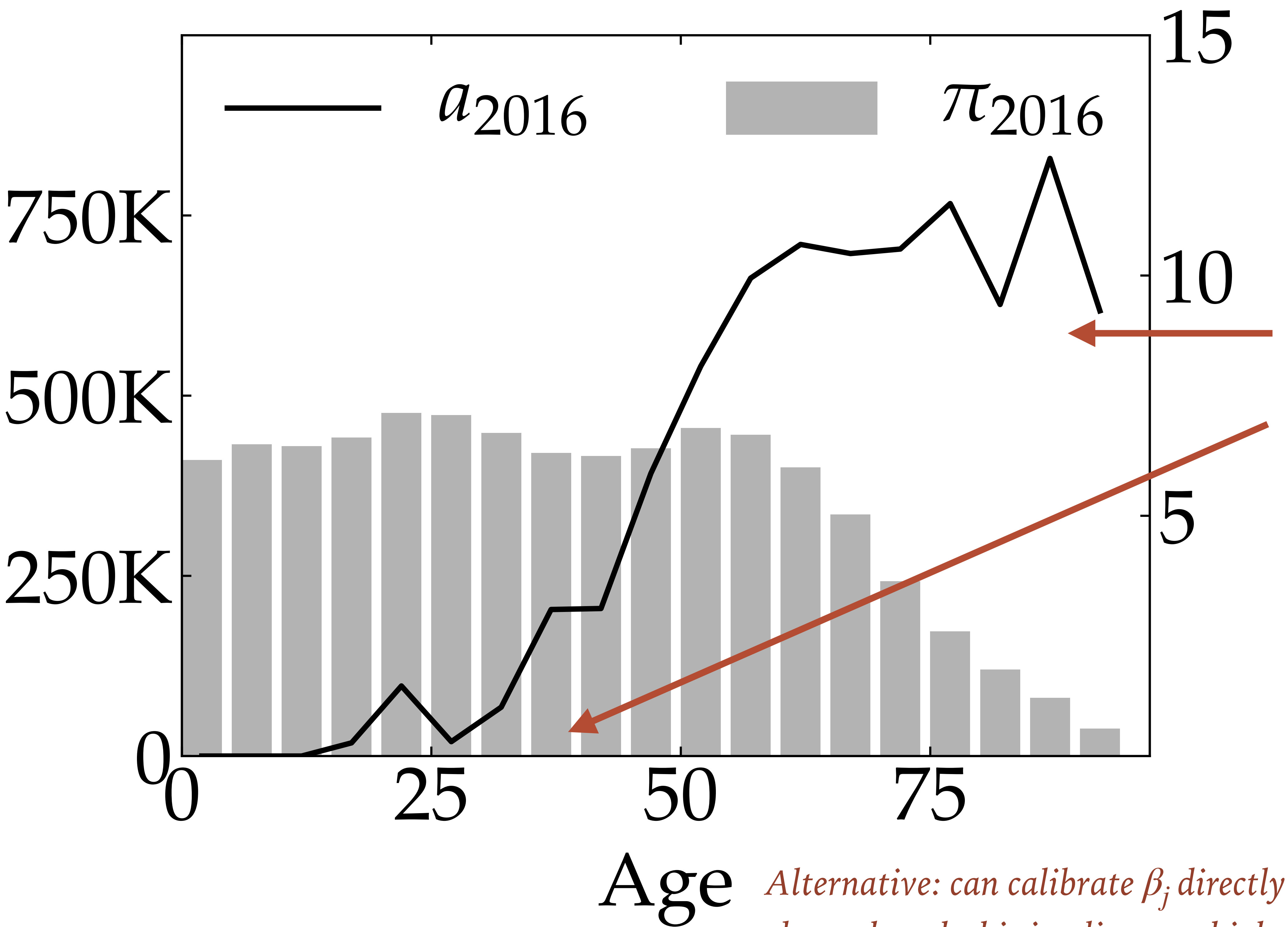
CONSUMPTION AND INCOME LIFECYCLE TRAJECTORIES



ASSET LIFECYCLE TRAJECTORY



NOT BAD, BUT NOT PERFECT COMPARED TO DATA



In aggregate, younger people don't really go into debt, and older people don't really draw down their assets

Alternative: can calibrate β_j directly to hit asset profile from data, though this implies very high consumption at old age.

TAKING STOCK: STRENGTHS AND WEAKNESSES OF THE MODEL

- Model lets us incorporate mortality risk and life-cycle patterns in income
  - In simple example, leads to life-cycle path of assets that looks roughly right, but misses some important dimensions

- What's missing from the model?
  - **Borrowing constraints and income risk**, and any other forces generating heterogeneity within cohort
  - **Bequests**—big saving motivation for older rich
  - **Social security system** providing government-funded income to old

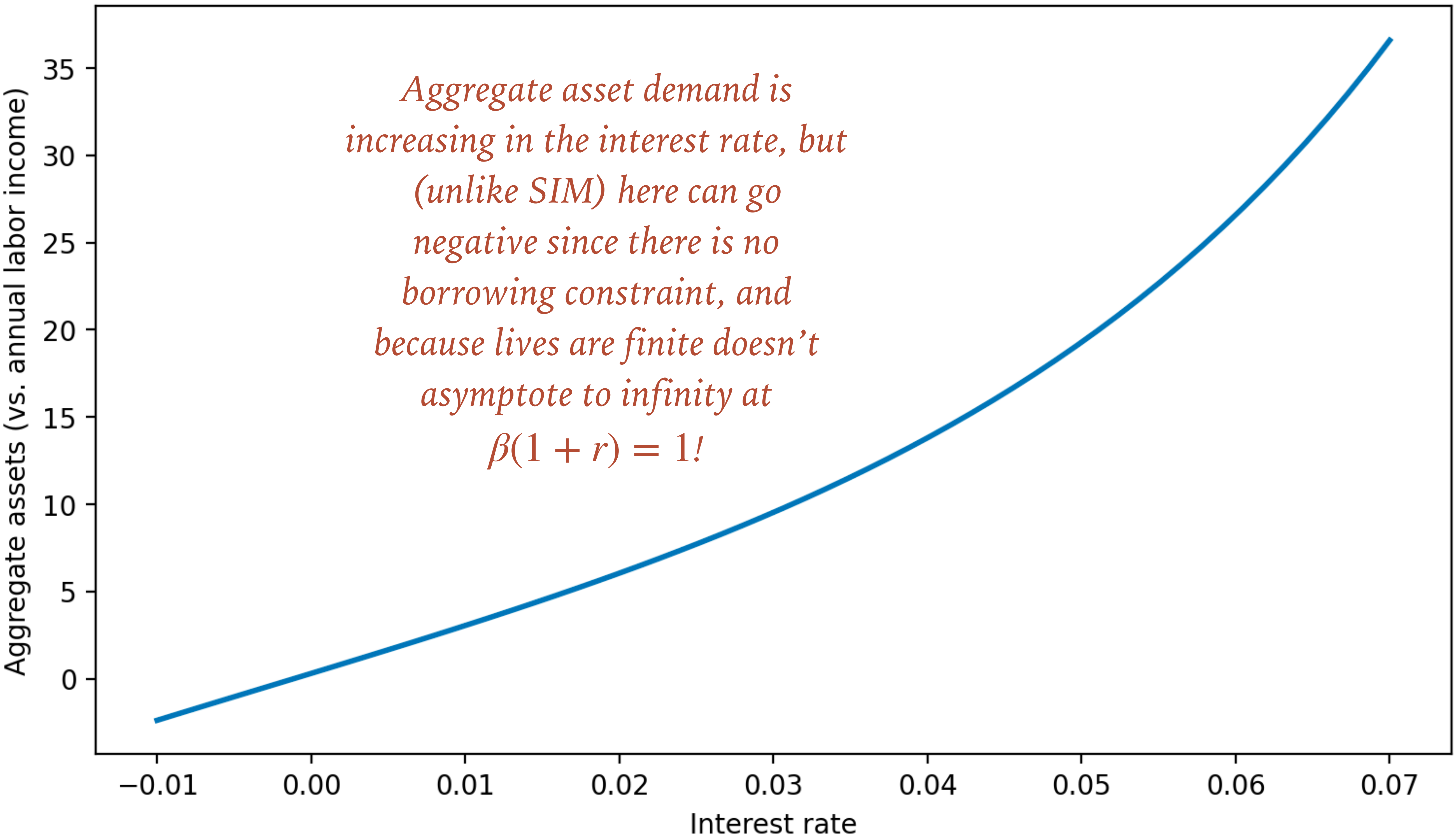
(Won't add in this lecture, but all these appear in Section 5 quantitative model in "Demographics, Wealth…" paper)

AGGREGATE CONSEQUENCES

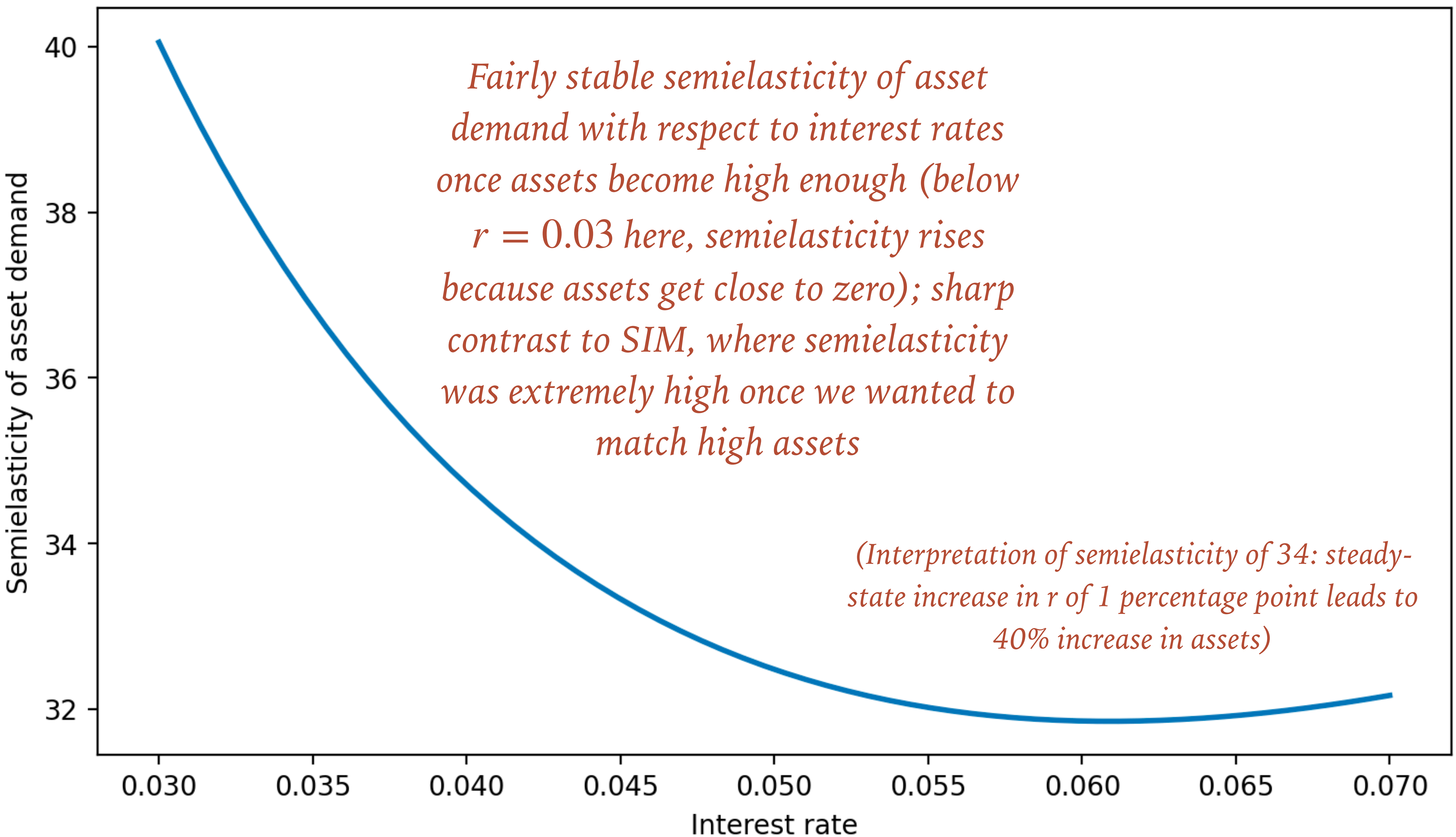
LOOK AT AGGREGATE STEADY-STATE MODEL

- We'll now proceed like we did with the standard incomplete markets model
- Think of this life-cycle model as existing in a world of "overlapping generations", where people of all ages live together
- Concretely, suppose that we are in a **"demographic steady state"**, where the number of "newborns" (age 0) grows at a constant rate g
- Then **steady-state age distribution** given by $\pi_j \propto (1+g)^{-j}\Phi_j$
  - More young in population when g is high, and vice versa
  - We'll use $g = 0$ as our baseline for simplicity

AGGREGATE ASSET DEMAND



HOW SENSITIVE ARE ASSETS TO INTEREST RATES?



HOW CAN WE UNDERSTAND THESE MAGNITUDES?

- Define Age_a as a random variable giving "at what age is a random dollar of assets in the economy held". Similar for Age_c.
- Result (AMMR): if $r = g = 0$, then locally we have:

$$e_r^d = \underbrace{\sigma \frac{C}{A} Var(Age_c)}_{\equiv \epsilon_r^{d,sub}} + \underbrace{\mathbb{E}[Age_c] - \mathbb{E}[Age_a]}_{\equiv \epsilon_r^{d,inc}}$$

- Intuition: $\epsilon_r^{d,sub}$ gives ***substitution effect***, and is proportional to C, EIS σ, and the variance of ages at which goods are consumed; latter gives the *scope for substitution*
- $\epsilon_r^{d,inc}$ gives ***income effect***: higher r means we earn more at the (older) ages when we hold assets, and we use this income to scale up consumption at all ages; on average, this cuts assets by $\mathbb{E}[Age_c] - \mathbb{E}[Age_a]$

IMPLEMENTING THIS CALCULATION FOR EXAMPLE CALIBRATION

$$e_r^d = \underbrace{\sigma \frac{C}{A} Var(Age_c)}_{\equiv \epsilon_r^{d,sub}} + \underbrace{\mathbb{E}[Age_c] - \mathbb{E}[Age_a]}_{\equiv \epsilon_r^{d,inc}}$$

$\sigma = 1,\ C/A = .135,\ Var(Age_c) = 400$

$\epsilon_r^{d,sub} = 54$

$\mathbb{E}[Age_c] = 59.3, \quad \mathbb{E}[Age_a] = 67.6$

$\epsilon_r^{d,inc} = -8.3$

$\epsilon_r^d = 45.7$

(vs. actual $\epsilon_r^d = 40.1$)

(Note: generalized version of exact result holds whenever r-g=0, where g includes pop & tech growth; in practice, can be pretty accurate.)

*Not exact because we don't have $r = 0$ here (instead $r = 0.03$), but roughly right. Clarifies the two opposing forces: a powerful **substitution effect** that scales with the EIS and variance of consumption over lifecycle, and a generally negative but weaker **income effect**, as people need to save less for retirement when they'll earn a higher return on their assets.*

APPLICATION: EFFECT OF CHANGES IN POPULATION GROWTH ON INTEREST RATES

APPLICATION: CHANGE IN POPULATION GROWTH RATE (FERTILITY)

- Exactly as in the Aiyagari model, if we close the model by assuming that all assets are capital, we get an equilibrium condition of the form

$$a(r, g) = \frac{k(r)}{w(r)}$$

- Here, $a(r, g)$ is aggregate household asset demand divided by labor income, and we make its dependence on population growth explicit by including g as an argument
- Same results as with Aiyagari, e.g. if shock to g then:

$$dr = -\frac{\epsilon_g^d dg}{\epsilon_r^d + \epsilon_r^s}$$

WHAT'S DIRECT EFFECT OF G ON ASSETS?

- A change in population growth g doesn't directly change assets or labor earnings at any age, it just changes the composition of the population
- Can show that we have simply:

$$\begin{aligned}\epsilon_g^d &= -(1+g)^{-1}(\mathbb{E}[Age_a] - \mathbb{E}[Age_y]) \\ &\approx -(67.6 - 47.4) = 20.2\end{aligned}$$

- Intuitively, why is this? A fall in g shifts population distribution toward older ages, and to the extent assets are held by older people than labor income is, this raises the asset-to-labor-income ratio

CONCLUSION: OVERALL EFFECT OF G DECLINE?

- We calculate (using $\epsilon_r^s = 1/(r+\delta) = 1/(0.03+0.05)$):

$$\frac{dr}{dg} = -\frac{\epsilon_g^d}{\epsilon_r^d + \epsilon_r^s} = \frac{20.2}{40.1 + 12.5} \approx 0.384$$

- So here, a decline in the population growth rate from 0% to -1% will cause a decline of about 40 basis points in real interest rates
- Growth of -1% corresponds to shrinking by 26% each 30-year generation (or total fertility of $2.1/e^{.3} \approx 1.55$), similar to many developed countries today—but not nearly as low as the lowest (e.g. Korea)

SENSITIVITY TO PARAMETERS

- If we keep the same calibration but change the EIS σ to 0.5, perhaps a more reasonable value in the literature:

$$\epsilon_g^d = -\left(\mathbb{E}[Age_a] - \mathbb{E}[Age_y]\right)$$
$$\approx -(74.9 - 47.4) = 27.5$$

- Now we're closer to a percentage-point effect on r:

$$\frac{dr}{dg} = -\frac{\epsilon_g^d}{\epsilon_r^d + \epsilon_r^s} = \frac{27.5}{21.0 + 12.5} \approx 0.82$$

- Why? Mainly because smaller σ shrinks ϵ_r^d, but also assets more disproportionately held by old (less substitution toward consumption when old). These larger effects fairly reasonable.