Econ 411-3 Problem Set 2

Due by 11:59 PM, Friday, April 26[1]

In this problem set, you will build upon the life-cycle analysis we did in lecture 6. You will want to look at, and build upon, the underlying code in `lecture 6 figures.ipynb` and `life_cycle.py`, posted on Canvas.

1. For lecture 6, we calibrated a life-cycle model by specifying the survival Φ_j, the age-specific income y_j, and other parameters (r, σ), and then assuming geometric discounting β_j. This, via the household Euler equation and budget constraint, implied consumption c_j and assets a_j.

Continue to use the same Φ_j and y_j as in lecture 6, and also calibrate $r = 0$ and $\sigma = 1$. But now, rather than specifying an exogenous β_j, let's calibrate the model to target a reasonable path for assets. In particular, let's define the function

$$f(j) \equiv \frac{1}{1 + e^{-0.14\cdot(j-30)}}$$

and then calibrate assets at each j to be proportional to this (minus its value at the initial age $j = 0$):

$$a_j \propto f(j) - f(0)$$

Scale assets so that average assets over the lifecycle equal 5, i.e. $\sum_j \Phi_j a_j / \sum_j \Phi_j = 5$. (Since we have normalized $\sum_j \Phi_j y_j / \sum_j \Phi_j = 1$, this is tantamount to saying that the aggregate asset-to-labor-income ratio is 5.) Plot assets by age up to age 100 (with the convention that $j = 0$ is age 20, so that $j = 80$ is age 100).

2. Back out consumption c_j at each age j from the budget constraint

$$c_j + \phi_j a_{j+1} = y_j + (1 + r)a_j$$

using the calibrated a_j, y_j, and ϕ_j, and $r = 0$. Verify that your answer satisfies the survival-weighted present-value budget constraint. Plot consumption by age up to age 100. Do you notice anything interesting? What features of the calibration are producing this outcome?

Finally, back out the utility shifters β_j that rationalize this path of consumption from the intertemporal Euler equation, $\beta_j c_j^{-1/\sigma} = (1 + r)^{k-j} \beta_k c_k^{-1/\sigma}$, given $r = 0$ and $\sigma = 1$.

3. We now want to derive the *substitution* and *income* components of the consumption and asset response to r.

Recall that the life-cycle consumption and asset plan is determined fully by two equations: the intertemporal Euler equation and the intertemporal budget constraint. The real interest rate r appears in both. Let's call the r that appears in the Euler equation r^{sub}, and the r that appears in the budget constraint r^{inc}. We define the *substitution effect* of r as the effect from perturbing only r^{sub}, and the *income effect* of r as the effect from perturbing only r^{inc}.[2] The sum of the two is the overall effect of r.

[1] You can work in groups of up to four; you only have to submit once per group, but remember to list all members of the group when submitting. Please email solutions to Jose Lara (`joselara@u.northwestern.edu`), including whatever code you used to produce them. You may want to do much or all of the problem set in the form of a Jupyter notebook. For this problem set, you are free to reuse any code that has been posted for the lectures on Canvas as part of your solution.

[2] At least for this case, I believe that this decomposition coincides with the decomposition between substitution and income effects that we'd define in a micro class, but we won't try to prove that.

First, calculate the derivatives of consumption c_j with respect to r^{sub} and r^{inc} from age 20 to 100, and plot these "substitution" and "income" effects, plus the total effect. Do the same for assets a_j. (You can potentially do this in two ways. First, you could modify the code in `life_cycle.compute_lifecycle` so that it takes in separate "substitution" and "income" interest rates r, and then perturb them separately. Second, you could use the unmodified function, making use of the observation that the effect of r in the Euler equation is equivalent to a certain change in the β_j. In either case, you will probably want to compute numerical derivatives, e.g. where for a function g we approximate $g'(x) = \frac{g(x+h)-g(x-h)}{2h}$ for a small h like $h = 10^{-5}$.)

Next, aggregate the effects on assets and divide by A to obtain the substitution and income asset demand semielasticities, $\epsilon_r^{d,sub}$ and $\epsilon_r^{d,inc}$, along with the overall semielasticity $\epsilon_r^d \equiv \epsilon_r^{d,sub} + \epsilon_r^{d,inc}$. Assume that the population growth rate is $g = 0$, so that the population distribution is proportional to the survival function: $\pi_j \propto \Phi_j$.

Finally, verify that $\epsilon_r^{d,sub}$ and $\epsilon_r^{d,inc}$ exactly agree[3] with the formulas in the slides, namely $\epsilon_r^{d,sub} = \sigma \frac{C}{A} \text{Var}(Age_c)$ and $\epsilon_r^{d,inc} = \mathbb{E}[Age_c] - \mathbb{E}[Age_a]$, where Age_a and Age_c are random variables giving "at what age is a random dollar of assets/consumption in the economy held". Holding the age distributions of consumption and assets fixed,[4] how low does the elasticity of intertemporal substitution σ need to be for the overall semielasticity ϵ_r^d to become negative, so that asset demand slopes the "wrong" way with respect to r?

4. Now let's consider the steady-state effects of changes in the population growth rate g. Define $a(r, g)$ to be aggregate asset demand divided by aggregate labor income. This can be calculated from our life-cycle model as

$$a(r, g) = \frac{\sum_j \pi_j(g) a_j(r)}{\sum_j \pi_j(g) y_j} \tag{1}$$

where we write assets a_j as a function of r, and the population distribution π_j as a function of g (reflecting $\pi_j \propto (1+g)^{-j}\Phi_j$), and hold income at each age y_j fixed.[5]

We already have calculated $\epsilon_r^d \equiv \frac{\partial \log a}{\partial r}$ above. Now, calculate

$$\epsilon_g^d \equiv \frac{\partial \log a}{\partial g}$$

around our initial steady state of $r = 0$ and $g = 0$, and verify that the equation from the slides, $\epsilon_g^d = -(\mathbb{E}[Age_a] - \mathbb{E}[Age_y])$, holds.

5. Finally, let's calculate the overall first-order effect of a shock to population growth g, using the formula from the slides

$$dr = -\frac{\epsilon_g^d dg}{\epsilon_r^d + \epsilon_r^s} \tag{2}$$

[3] Up to some small numerical differentiation error, which should probably be smaller than 0.001.

[4] This is a slightly different exercise from what I did in class, where I changed σ and then held the β_j fixed while I recalculated c_j and a_j. Here, we're assuming that as we change σ, a_j and c_j (and therefore the expressions involving them) stay fixed, with β_j adjusting in the background to rationalize them. The idea is that a_j is our calibration target (and it implies c_j via the budget constraint), and we want to hold it fixed while we investigate how comparative statics vary with σ.

[5] In general equilibrium, the wage w will change with r, but this causes both a_j and y_j to scale up by the same factor, which cancels out in this ratio, which is why we ignore it.

where ϵ_r^s is the semielasticity of normalized asset supply $k(r)/w(r)$, which for Cobb-Douglas (as we derived in lecture 4) is just $1/(r+\delta)$. Set $\delta = 0.08$ and continue using $r = 0$.

Using the analytical formula for $\epsilon_r^d = \epsilon_r^{d,sub} + \epsilon_r^{d,inc}$, apply (2) as the elasticity of substitution σ varies from 0.1 to 1.5 to see how dr/dg depends on σ and plot the results.

Also plot the effect on the asset-income ratio, $da/dg = -\epsilon_r^s dr/dg = \frac{\epsilon_r^s}{\epsilon_r^d + \epsilon_r^s}\epsilon_g^d$, over the same range of σ. Comment on the results.