Optimal Long-Run Fiscal Policy with Heterogeneous Agents*

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Abstract

We introduce a new method for characterizing the steady state of dynamic Ramsey problems, building on the dual approach to optimal taxation. Applying this method to standard calibrations of heterogeneous-agent models à la Aiyagari (1995), we find that in many cases Ramsey steady states do not exist, with our results suggesting that long-run immiseration is optimal instead. When Ramsey steady states do exist, they are associated with optimal long-run labor income taxes close to 100%. We show that these conclusions are related to strong anticipatory effects of future tax changes.

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1 Introduction

Models of household behavior with uninsurable idiosyncratic income risk à la Bewley-Aiyagari-Huggett—which we call "heterogeneous-agent models" in this paper—have experienced enormous success over the past thirty years. At the micro level, these models are able to match individual behavior along a number of dimensions, including buffer-stock savings (Zeldes 1989, Deaton 1992, Carroll 1997), life-cycle consumption and income profiles (Gourinchas and Parker 2002, Storesletten, Telmer and Yaron 2004), contemporaneous and intertemporal MPCs (Kaplan and Violante 2014, Auclert, Rognlie and Straub 2024a), and labor supply choices (Blundell and MaCurdy 1999). At the macro level, these models are at the heart of modern analyses of growth and inequality (Castañeda, Díaz-Giménez and Ríos-Rull 2003) and studies of business cycles and stabilization policies (Krusell and Smith 1998, Kaplan, Moll and Violante 2018, Auclert, Rognlie and Straub 2020).

Given the prevalence of heterogeneous-agent models in macroeconomics today, surprisingly little is known about their normative properties. How should a planner trade off capital and labor taxation? How much debt should a planner use to finance the government, taking into account the self-insurance benefits to households? How progressive should taxes be, when the usual equity-efficiency trade-off is mitigated by precautionary saving? Does higher inequality imply a higher optimal level of public debt? Answers to these questions have remained elusive in large part because of the computational complexity of heterogeneous-agent models, especially once they are embedded into an optimal policy problem.

In this paper, we propose a new method to solve for the optimal long-run fiscal policy in heterogeneous-agent economies. Our method allows us to characterize the *Ramsey steady state (RSS)* of heterogeneous-agent models—the long-run steady state of the optimal full-commitment Ramsey plan. Ramsey steady states have been a central object of interest in optimal taxation for decades, going back to the early work of Chamley (1986) and Judd (1985), which characterized Ramsey steady states in representative-agent and two-agent models, respectively.[1] Our method, which builds on the dual approach to optimal taxation, exploits recent advances expressing macroeconomic models in the "sequence space" (Auclert, Bardóczy, Rognlie and Straub 2021). It delivers an intuitive RSS optimality condition that involves interpretable and potentially estimable *discounted elasticities*, generalizing related objects found in Piketty and Saez (2013) and Straub and Werning (2020).

We evaluate our RSS optimality condition numerically for standard parameterizations of heterogeneous-agent models. Our main finding is that Ramsey steady states in this class of models tend to involve extremely large tax rates. In many parameterizations with balanced growth preferences, a Ramsey steady state doesn't exist at all, with our results suggesting the optimality of long-run immiseration instead—labor taxes approaching 100% and real consumption falling to zero. In other parameterizations, a Ramsey steady state may exist, but it involves near-immiseration tax rates, generally above 90%. We find the only versions of the model that lead to reasonable Ramsey steady states far from immiseration are those with non-balanced growth preferences and no wealth

[1]See Chari and Kehoe (1999) for a review, as well as Straub and Werning (2020) for a recent qualification of this work.

effects on labor supply, as with "GHH" preferences (Greenwood, Hercowitz and Huffman 1988). This confirms some of the results from an earlier literature that has studied the RSS in models with GHH preferences (Aiyagari 1995, Acikgoz, Hagedorn, Holter and Wang 2018, LeGrand and Ragot 2023).

Model and RSS condition. We begin our exploration of Ramsey steady states in a baseline model à la Aiyagari (1995), modified in two ways: we allow for general household preferences, and assume production is linear in labor, abstracting away from capital at first to ease exposition. We assume the social planner can fully commit to paths of proportional labor income taxes and public debt at date 0. We allow for a general social discount factor, though our main focus is on the case where social and private discount factors coincide.

In this economy, aggregate household behavior can be summarized by *sequence-space functions*, that is, we can write aggregate household asset demand A_t at date t as a function of the entire sequences of after-tax interest rates and wages $A_t = \mathcal{A}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right)$; likewise, we can write aggregate labor supply N_t at date t as $N_t = \mathcal{N}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right)$. Similar sequence-space functions have proved very useful in the literature to perform decompositions of the equilibrium effects of policies (e.g. Kaplan et al. 2018, Farhi and Werning 2019), as well as to solve and estimate heterogeneous-agent models (Auclert et al. 2020, 2021, 2024a). We show that competitive equilibrium behavior is summarized by a single set of implementability constraints involving these sequence-space functions. This makes the dual approach to the optimal policy problem particularly tractable.

Our main theoretical result is a set of necessary optimality conditions that have to hold at the Ramsey steady state of our economy, in addition to the government budget constraint. We first derive a single optimality condition by assuming that the multiplier λ_t on our implementability condition converges. This condition is best thought of as equalizing the planner's benefits and costs of providing additional liquidity (raising r, benefiting households) financed by higher labor taxes (lowering w, hurting households). It equates the *liquidity benefit* of higher interest rates to the disincentive effects of lower wages on *labor supply* as well as the costs from *redistribution* of resources from the average worker to the average saver. We also consider the case where the multiplier on the implementability condition diverges $\lambda_t \to \infty$, in the spirit of Straub and Werning (2020). For this case, we derive a pair of optimality conditions, which involve three terms with similar interpretation.

Our RSS optimality conditions all involve *discounted elasticities* of sequence-space functions, such as the response $\epsilon^{N,w}$ of the present value of labor supply to a fully anticipated one-time increase in wages. These elasticities, which are potentially estimable in micro data, are "sufficient statistics" for household behavior in our setting. To check if a candidate steady state is an RSS, these discounted elasticities must be computed at the candidate steady state. We provide a method for doing so efficiently and accurately.

Immiseration and near-immiseration. We proceed by evaluating our optimality conditions numerically in common parameterizations of heterogeneous-agent models. We begin with standard log-separable preferences and a conventional calibration to the U.S. economy. We check the optimality condition for all possible steady states of the model, finding that the benefit from a marginal, fully-anticipated increase in interest rates and labor income taxes is always strictly positive. Thus, a Ramsey steady state cannot exist. Our results suggest that the economy instead tends towards immiseration, with labor taxes approaching 100% and real consumption converging to zero. We identify the discounted labor supply elasticity $\epsilon^{N,w}$ as the main force towards immiseration in the model: in any candidate steady state, this elasticity is *negative*, implying that greater labor taxation actually *increases* the present value of labor supply. As agents rationally anticipate greater future labor taxes, they raise their hours in the present. This turns the traditional labor supply cost of high labor taxation into a benefit.

We conduct an extensive numerical investigation of the robustness of our immiseration finding to the parameters of the model. We find that, as long as preferences are consistent with balanced growth as in King, Plosser and Rebelo (1988), there is either a Ramsey steady state very close to immiseration (labor taxes above 90%) or no Ramsey steady state at all, with results pointing to immiseration. This is true for a variety of different income processes, for different assumptions on initial government debt or government spending, and for models with lump-sum transfers and progressive taxes.

We also consider an economy with capital and capital income taxes, as in Aiyagari (1995). We derive modified RSS optimality conditions for this setting and show that, again, no Ramsey steady state exists, with results consistent with long-run immiseration. In addition, our results suggest that Lagrange multipliers diverge, breaking Aiyagari (1995)'s landmark modified golden rule result for capital accumulation in the RSS.

Non-balanced growth preferences and alternative household models. We also consider robustness of our findings to the specification of preferences. A common alternative to balanced-growth preferences are additively separable and GHH preferences. For additively separable preferences with an elasticity of intertemporal substitution (EIS) different from 1, we can find one or more candidate Ramsey steady states, though all are very close to immiseration with labor taxes near 100% for standard values of the EIS. GHH preferences also give rise to one or more candidate Ramsey steady states since wealth effects on labor supply are entirely absent with these preferences. This is consistent with Acikgoz et al. (2018)'s findings.

We end our paper by discussing several alternative models of household behavior. We find that bond-in-utility models—often regarded as "tractable" versions of heterogeneous-agent models—have conceptually similar predictions for Ramsey steady states as our full-blown Aiyagari model. Indeed, we also find immiseration in a standard log-separable version of such a model. We show how our optimality conditions are also useful in understanding why standard overlapping generations models and models with alternating income states (Woodford 1990) generally do not lead to immiseration.

Taking stock and limitations. Our finding of the optimality of (near-)immiseration in a large class of otherwise very reasonable heterogeneous-agent models can be interpreted in two ways. One interpretation is that the immiseration result should lead us to modify the planning problem, by choosing a higher social discount factor or by assuming limited commitment. While we are sympathetic to this interpretation, one should note that households themselves will still *desire* long-run immiseration, even if the planner prefers not to, or cannot, deliver it to them.

The second interpretation is that a negative elasticity $\epsilon^{N,w}$ likely has the wrong sign relative to what empirical estimates would suggest. If so, it would be interesting to extend our analysis to allow for richer models of household labor supply or directly dampen the anticipatory effects of wage changes—similar to dampened anticipation effects in García-Schmidt and Woodford (2019) or Gabaix (2020). This could flip the sign of $\epsilon^{N,w}$ to be positive, making immiseration no longer optimal.

Our approach also has some limitations. Just like much of the previous literature, we only derive a necessary first order condition for the Ramsey steady state, not a second order condition. This is sufficient for the main results in this paper—the absence of a Ramsey steady state without (near-)immiseration—but may not be enough for other questions. We also do not solve for transitional dynamics, which could be especially interesting in economies with long-run immiseration. Given the wealth of materials already in this paper, we view this as a separate project.

Literature. Our paper is most closely related to the literature studying Ramsey taxation in heterogeneous agent models, specifically Aiyagari (1995), Acikgoz et al. (2018), Chien and Wen (2022), Dyrda and Pedroni (2023), and LeGrand and Ragot (2023). Relative to these papers, ours develops a new method to characterize Ramsey steady states using the dual, and demonstrates how in many standard cases (though not all) immiseration is optimal. Since it is easiest to relate to these papers in detail after presenting our results, we dedicate section 9 to this comparison.

Our paper is also related to the classic paper by Aiyagari and McGrattan (1998), which assumes an infinitely patient planner, that is, a planner that only cares about the long-run steady state (henceforth the *optimal steady state, OSS)*, ignoring the transition it takes to get there. We also find that the OSS always exists, but our focus is instead on the case where the social and private discount factors coincide.[2] Boar and Midrigan (2022) solve a planning problem that does take transitions into account, but restrict attention to constant tax rates. Aguiar, Amador and Arellano (2021) use a different welfare criterion (robust Pareto improvements) which differs from the Ramsey steady state analyzed here.

Dávila, Hong, Krusell and Ríos-Rull (2012) consider the optimality of individual saving behavior in a heterogeneous-agent economy. Their constrained efficient allocation corresponds to that chosen

[2] A large and influential literature following Aiyagari and McGrattan (1998) uses a similar approach based on the OSS, see e.g. Conesa, Kitao and Krueger (2009).

by a planner that is able to levy individual-specific capital taxes or subsidies, whose revenue is then rebated lump-sum back to the individual agents. We focus, instead, on capital and labor tax schedules that are the same for all agents, and allow the government to borrow and spend. Dávila and Schaab (2023) characterize the Ramsey steady state of a heterogeneous-agent New-Keynesian model if the planner controls monetary (but not fiscal) policy. Bhandari, Evans, Golosov and Sargent (2021) allow the planner to choose both monetary and fiscal policy but only analyze the planner's response to shocks without characterizing the Ramsey steady state. Werning (2007) and Bassetto (2014) study the Ramsey steady state with household heterogeneity but without idiosyncratic income risk. Finally, there is a literature about immiseration results with endogenously incomplete markets (e.g. Thomas and Worrall 1990, Atkeson and Lucas 1992, Phelan 1995, Farhi and Werning 2007). Our paper shows that similar dynamics are possible with exogenously incomplete markets.

Layout. The paper proceeds as follows. Section 2 sets up the model and derives the implementability condition using sequence-space functions. Section 3 derives the RSS optimality conditions with and without converging multipliers. Sections 4 presents our immiseration result for a standard model with balanced-growth preferences, section 5 considers more general parameterizations, and section 6 considers alternative preferences. Section 7 adds capital to the model, and section 8 considers alternative models of household behavior. We review the literature in section 9 and conclude in section 10. All proofs are relegated to the appendix.

2 Heterogeneous-agent economy

Our economy is a standard Aiyagari (1994, 1995) economy, with flexible labor supply and a general utility function. The only simplification we make in the first part of the paper is that we study an economy without capital. However, as we show in section 7, the main conclusions we draw here carry over to an economy with capital and capital taxes. Time is discrete, $t = 0, 1, 2, \ldots$, and there is no aggregate risk. All agents have perfect foresight with respect to aggregate variables.

2.1 Households

There is a continuum of households, labeled by $i \in [0, 1]$. At the beginning of each period, household i draws idiosyncratic productivity e_{it} from a stochastic process described by a positive and recurrent finite-state Markov chain; e_{it} is iid across households, with a mean normalized to 1. We further allow for an end-of-period productivity shock $\epsilon_{it} > 0$ that is iid over time and across households, and drawn from a smooth distribution with density $\vartheta(\epsilon)$ and mean 1. ϵ_{it} is realized after consumption and labor supply are chosen. We allow for ϵ_{it} to ensure that the wealth distribution predicted by the model is smooth, which will help with our theoretical results below. For all intense and purposes, ϵ_{it} can be thought of as being arbitrarily close to 1. In all our numerical exercises, ϵ_{it} is set to 1.

Household i's date-0 expected utility is given by

$$\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_{it}, n_{it}) \right] \tag{1}$$

Here, $\beta \in (0,1)$ is the private discount factor, c_{it} denotes consumption and n_{it} labor supply. $u(c,n)$ is a general per-period utility function satisfying the usual Inada conditions.[3] Households are able to self-insure against their idiosyncratic productivity shocks by holding assets a_{it}, earning an interest rate r_t between periods $t-1$ and t. Their budget constraint is given by

$$c_{it} + a_{it} = (1 + r_t)\, a_{it-1} + w_t e_{it} \epsilon_{it} n_{it} \tag{2}$$

where $w_t > 0$ is the date-t after-tax wage per effective unit of labor. Households are subject to a standard borrowing constraint

$$a_{it} \geq 0 \tag{3}$$

We also allow for an upper bound on assets, $a_{it} \leq \overline{a}$ where $\overline{a} \in (0, \infty]$. For our theoretical results, we assume $\overline{a} < \infty$ is some arbitrarily large positive constant. For our numerical results, we set $\overline{a}$ sufficiently high that it is not binding. Taken together, households choose $\{c_{it}, n_{it}, a_{it}\}$ in order to maximize (1) subject to (2) and (3).

As shown in Auclert et al. (2024a), aggregate (partial equilibrium) household behavior in this economy can be expressed entirely as a function of the sequences of interest rates and wages, $\{r_t, w_t\}_{t=0}^{\infty}$—the two prices in the economy. Conditional on $\{r_t, w_t\}_{t=0}^{\infty}$, households can solve for their consumption policies $c_t^*(e, a_-)$, labor supply policies $n_t^*(e, a_-)$, and savings policies $a_t^*(e, a_-)$.[4] These policies then perfectly determine the evolution of the wealth distribution $\Psi_t(e, a_-)$, starting from some arbitrary given initial distribution $\Psi_0(e, a_-)$. Date-t aggregate (partial equilibrium) household asset demand can then be written as

$$A_t = \mathcal{A}_t \left(\{r_s, w_s\}_{s=0}^{\infty} \right) \equiv \int a_t^*(e, a_-)\, d\Psi_t(e, a_-) \tag{4}$$

Aggregate (partial equilibrium) effective labor supply can be written as

$$N_t = \mathcal{N}_t \left(\{r_s, w_s\}_{s=0}^{\infty} \right) \equiv \int n_t^*(e, a_-) d\Psi_t(e, a_-) \tag{5}$$

Aggregate consumption can be written as

$$C_t = \mathcal{C}_t \left(\{r_s, w_s\}_{s=0}^{\infty} \right) \equiv \int c_t^*(e, a_-) d\Psi_t(e, a_-) \tag{6}$$

[3] That is: $u : (0,\infty) \times [0,\infty) \to \mathbb{R}$ is twice differentiable and strictly concave, with $\lim_{c \to 0} u_c(c,n) = \infty$, $\lim_{c \to \infty} u_c(c,n) = 0$ for any $n \geq 0$; and $\lim_{n \to 0} u_n(c,n) = 0$, $\lim_{n \to \infty} u_n(c,n) = \infty$ for any $c > 0$.

[4] We construct $a_t^*(e,a_-)$ as the average saving of households in state (e, a_-) at the beginning of period t, that is, $a_t^*(e,a_-) = \int \tilde{a}_t^*(e, a_-, \epsilon)\, \vartheta(\epsilon) d\epsilon$, where $\tilde{a}_t^*(e, a_-, \epsilon) = (1 + r_t)\, a_- + w_t e \epsilon n_t^*(e, a_-) - c_t^*(e, a_-)$.

Finally, utilitarian flow utility can be written as

$$U_t = \mathcal{U}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right) \equiv \int u(c_t^*(e,a_-), n_t^*(e,a_-))d\Psi_t(e,a_-) \tag{7}$$

Each one of these four "curly" functions, $\mathcal{A}, \mathcal{N}, \mathcal{C}$, and $\mathcal{U}$, is a *sequence-space function*, in that it maps the two sequences of prices into sequences of aggregate household behavior and household utility.

We make the following main assumption:

Assumption 1 (Stationarity). *For any $r < 1/\beta - 1$ and $w > 0$, there exists a unique and globally stable steady state wealth distribution $\Psi(e, a_-)$.*

Assumption 1 is a basic assumption that holds in all heterogeneous-agent models. If stationarity is not satisfied, as in a standard representative-agent model, the results in this paper do not apply. With stationarity, the derivatives of the four curly functions $\mathcal{A}, \mathcal{N}, \mathcal{C}, \mathcal{U}$ have specific shapes, namely that of *β-quasi-Toeplitz matrices.*

Definition 1. An infinite matrix $\mathbf{M} = [M_{t,s}] \in \mathbb{R}^{\mathbb{N}\times\mathbb{N}}$ is a *β-quasi-Toeplitz matrix with symbol vector* $\mathbf{a} = \{a_t\} \in \mathbb{R}^{\mathbb{Z}}$ if:

1. for any $t, s \geq 0$ we have

$$\lim_{u\to\infty} M_{t+u,s+u} = a_{t-s} \tag{8}$$

2. $M_{t,s}$ decays exponentially away from the diagonal, that is, there exists a $\psi \in (0,1)$ such that

$$|M_{t,s}| \leq \begin{cases} D\psi^{t-s} & t-s \geq 0 \\ D\left(\beta\psi\right)^{-(t-s)} & t-s < 0 \end{cases} \tag{9}$$

A Toeplitz matrix is a matrix whose elements are constant along all sub-diagonals, that is, $M_{t,s}$ only depends on $t - s$ (e.g. Böttcher and Silbermann 2006). A quasi-Toeplitz matrix is one which has this property approximately, for large t, s, as in (8) (e.g. Bini, Massei and Robol 2019). A *β-quasi-Toeplitz matrix* is one in which entries $M_{t,s}$ decay at least at speed β as we make s much larger than t, as in (9), and at least exponentially as we make t much larger than s.

The reason β-quasi-Toeplitz matrices are useful for our analysis is that, as it turns out, they exactly embody the structure of derivatives of sequence-space functions of stationary economic models, such as $\mathcal{A}, \mathcal{N}, \mathcal{C}, \mathcal{U}$ for our heterogeneous-agent household side. Intuitively, $M_{t,s}$ with $t < s$ will capture anticipatory effects; those are naturally dampened with an additional β (see also Auclert, Rigato, Rognlie and Straub 2024b), relative to the lagged effects in elements $M_{t,s}$ with $t > s$.

In appendix **??** we prove the following (highly non-trivial) result about the sequence-space functions $\mathcal{A}, \mathcal{N}, \mathcal{C}, \mathcal{U}$.

Proposition 1. *Each one of the curly functions $\mathcal{X} \in \{\mathcal{A}, \mathcal{N}, \mathcal{C}, \mathcal{U}\}$ is a well-defined continuous function w.r.t. the sup norm. Evaluated at convergent sequences $\{r_s, w_s\}_{s=0}^{\infty}$ with $r_s \to r^* < 1/\beta - 1$ and $w_s \to w^* > 0$,* *(i)* $\mathcal{X}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right)$ *converges to a limit* $\mathcal{X}^{ss}(r^*, w^*)$ *that only depends on the limits* r^*, w^*.

(ii) $\mathcal{X}$ *is Fréchet-differentiable in each of its two arguments, with* β*-quasi-Toeplitz derivatives* $\mathbf{X}^{(r)}\left(\{r_s, w_s\}_{s=0}^{\infty}\right) : \ell^\infty \to \ell^\infty$ *and* $\mathbf{X}^{(w)}\left(\{r_s, w_s\}_{s=0}^{\infty}\right) : \ell^\infty \to \ell^\infty$ *whose symbol vector only depends on the limits* r^*, w^*.

Proposition 1 characterizes the derivatives (Jacobians) of the sequence-space functions $\mathcal{A}, \mathcal{N}, \mathcal{C}, \mathcal{U}$. It shows that each of the Jacobians is indeed a β-quasi-Toeplitz matrix. We will use this result extensively below in the derivation of our optimality conditions. We believe the result will be helpful for the emerging literature around heterogeneous-agent models in the sequence-space beyond this paper. The properties listed in the result are also true in the tractable overlapping generations and bonds-in-utility models we analyze in sections 8.2 and 8.3.

The limit $\mathcal{X}^{ss}$ introduced in property (i) characterizes the steady state behavior of the variable $\mathcal{X}$ as a function of the steady state interest rate and the wage. We write $\mathcal{A}^{ss}(r, w)$ for steady-state asset holdings as function of r and w; $\mathcal{N}^{ss}(r, w)$ for steady-state labor supply; $\mathcal{C}^{ss}(r, w)$ for steady-state consumption; and $\mathcal{U}^{ss}(r, w)$ for steady state flow utility.

We consider several parameterizations of household utility functions in this paper. Given that we are studying the long run, our main focus is on preferences that are compatible with balanced growth, à la King et al. (1988) (KPR), given by

$$u(c, n) = \frac{\left(ce^{-v(n)}\right)^{1-\sigma} - 1}{1-\sigma} \tag{10}$$

for some $\sigma > 0$, and $v(n) \geq 0$ denoting the disutility from labor supply. We focus in particular on the $\sigma = 1$ case, which delivers the standard log-separable utility function,

$$u(c, n) = \log c - v(n) \tag{11}$$

2.2 Production and government policy

Production is perfectly competitive and linear in effective labor. The unique output good at date t is produced using the technology

$$Y_t = N_t \tag{12}$$

This implies that the pre-tax wage w_t^* is equal to 1 at all times, $w_t^* = 1$.

The government is financing an exogenous amount of government spending $G > 0$ using a mix of proportional labor income taxes $\{\tau_t\}$ and government debt $\{B_t\}$. After-tax wages are therefore simply

$$w_t = 1 - \tau_t \tag{13}$$

and tax revenue from labor income taxation is given by $\tau_t N_t$. The government budget constraint is

$$G + (1 + r_t) B_{t-1} = B_t + \tau_t N_t \tag{14}$$

In appendix F.1, we discuss how our results carry over to environments with endogenous government spending.

2.3 Competitive equilibrium and implementability

We define equilibrium as follows.

Definition 2. A *competitive equilibrium* in our economy is a collection of quantities $\{Y_t, N_t, B_t, C_t, A_t\}_{t=0}^{\infty}$, tax rates $\{\tau_t\}_{t=0}^{\infty}$, and prices $\{r_t, w_t\}_{t=0}^{\infty}$, such that:

1. The after-tax wage is given by (13).
2. Households optimize given prices: aggregate consumption C_t is given by (6), aggregate assets A_t are given by (4), and aggregate effective labor supply N_t is given by (5).
3. Output is given by (12).
4. The government budget constraint (14) holds.
5. The asset market clears, $A_t = B_t$, and the goods market clears, $C_t + G = Y_t$.

A competitive equilibrium is a *steady state equilibrium* if all quantities, tax rates, and prices are constant.

Our approach builds on the dual approach to Ramsey taxation. This lets us derive simple implementability conditions, stated directly in terms of the sequence-space functions defined above.

Proposition 2 (Implementability). $\{r_t, w_t\}_{t=0}^{\infty}$ *are part of a competitive equilibrium if and only if*

$$\mathcal{C}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right) + G = \mathcal{N}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right) \tag{15}$$

or alternatively, if and only if

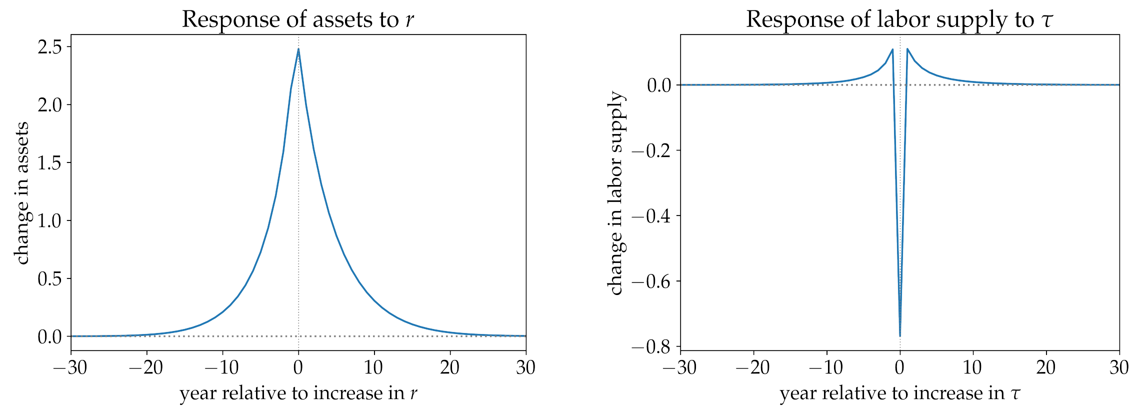
$$G + (1 + r_t)\,\mathcal{A}_{t-1}\left(\{r_s, w_s\}_{s=0}^{\infty}\right) = \mathcal{A}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right) + (1 - w_t)\,\mathcal{N}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right) \tag{16}$$

This result may appear surprising at first. How can the goods market clearing condition (15) be, on its own, sufficient for equilibrium? The reason is that we also have substituted optimal household behavior into this equation via sequence-space functions. In other words, the sequences of household consumption $\{C_t\}$ and labor supply $\{N_t\}$ that clear the goods market are the result of aggregated-up, optimizing household behavior. Setting government debt B_t equal to household optimal asset demand $\mathcal{A}_t\left(\{r_s, w_s\}\right)$ then naturally clears the asset market. The only remaining equilibrium condition is the government budget constraint, but this must hold by Walras' law. The logic behind the sufficiency of the government budget constraint (16) in proposition 2 is similar.

The fact that sequence-space functions imply a single implementability condition—either (15) or (16)—makes a dual approach to Ramsey taxation especially attractive. By contrast, a primal

Figure 1: Household responses to infinitely anticipated shocks

(a) Asset demand response to interest rate increase **(b)** Labor supply response to labor tax increase



approach would require a much larger number of implementability conditions, typically at least one for each agent in the economy (see e.g. Acikgoz et al. 2018).

2.4 Infinitely anticipated shocks and discounted elasticities

An important thought experiment in our analysis is that of an *infinitely anticipated shock* in partial equilibrium. Consider the households in our economy entering period $t = 0$ in a steady state, consistent with constant interest rate r and after-tax wage w. Then, imagine that the interest rate is announced to increase by some small amount, for a single period s, at some far-away date $s \gg 0$. Everything else remains the same. How does the time path of aggregate household asset demand A_t respond?

Using the sequence-space function for asset demand (4), we see that the date $t = s + h$ response, in percentage points of steady state assets, is given by the derivative

$$\frac{\partial \log \mathcal{A}_{s+h}}{\partial r_s} \tag{17}$$

where h runs from $-s$ to ∞. It turns out that for s sufficiently large, these derivatives become independent of s, and only depend on the time horizon h relative to the date of the anticipated shock (see Auclert et al. 2021, Auclert et al., 2024a.) For our baseline calibration below (see section 4.1), figure 1(a) plots the derivatives $\partial \log \mathcal{A}_{s+h}/\partial r_s$. We can see that households basically do not respond more than 30 years prior to the shock, then build up assets to earn the additional interest rate and subsequently decumulate assets back down to the steady state.

To give a second example, figure 1(b) plots the derivatives of labor supply to an increase in labor income taxes (equivalent to a decrease in after-tax wages)

$$-\frac{\partial \log \mathcal{N}_{s+h}}{\partial \log w_s} \tag{18}$$

with a large date s. Here also, we observe that households barely respond 30 years before the shock. Then, they slowly increase labor supply to offset the future income loss, sharply reduce their labor supply in the period of the tax increase due to a traditional substitution effect, and then return to working harder and offsetting the income loss. This anticipatory increase in labor supply to higher future labor income taxes will play a prominent role in this paper.

One way to summarize responses to infinitely anticipated shocks, such as the ones in (17) and (18), is to discount them relative to the period of the shock. For a general discount factor $\delta \in [\beta, 1]$, we define the following *discounted elasticities*

$$\epsilon^{A,r}(\delta) \equiv \lim_{s\to\infty} \sum_{h=-s}^{\infty} \delta^h \frac{\partial \log \mathcal{A}_{s+h}}{\partial r_s} \qquad \epsilon^{N,\tau}(\delta) \equiv -\epsilon^{N,w}(\delta) \equiv -\lim_{s\to\infty} \sum_{h=-s}^{\infty} \delta^h \frac{\partial \log \mathcal{N}_{s+h}}{\partial \log w_s} \tag{19}$$

We define discounted elasticities of all our other sequence space functions analogously: e.g. $\epsilon^{U,w}(\delta) \equiv \lim_{s\to\infty} \sum_{h=-s}^{\infty} \delta^h \frac{\partial \mathcal{U}_{s+h}}{\partial \log w_s}$, and similarly $\epsilon^{A,w}(\delta), \epsilon^{U,r}(\delta), \epsilon^{C,w}(\delta)$ and so on. Whenever we write a derivative with respect to tax rates τ, we interpret that as the negative of the elasticity with respect to after-tax wages, as in (19). Our discounted elasticities are similar to those introduced by Piketty and Saez (2013) and Straub and Werning (2020). We show in appendix B.1 that, because of assumption **??**, the discounted elasticities defined here are indeed well-defined in our model. Evaluating these elasticities numerically is central to our method; we provide an efficient algorithm for doing so in appendix C.

The discount factor δ is left general for the moment. It will later take the role of the social discount factor, so a natural baseline will be to assume it equals the private discount factor, $\delta = \beta$. For the numerical examples shown in figure 1, we find $\epsilon^{A,r}(\beta) \simeq 25$ and $\epsilon^{N,\tau}(\beta) \simeq 0.15$ when using the private discount factor. Notably, the discounted labor supply elasticity is positive in this case; this is due to the anticipatory increase in labor supply before the tax increase.

While we focus on a specific description of household behavior in this section—one based on heterogeneous agents and uninsurable idiosyncratic income risk—similar discounted elasticities can be defined and computed for any other stationary model of household behavior, including bonds-in-utility and life-cycle models. We explore alternative household sides in section 8.

3 Sequence-Space Approach to the Ramsey Steady State

We are now ready to set up the full-commitment Ramsey problem and derive our main formal result: a necessary condition characterizing the long-run steady state of the Ramsey plan, the *Ramsey steady state* (RSS). Throughout this section, we work with a general social discount factor $\delta \in [\beta, 1)$.

In the next section, we will also consider the limit $\delta \to 1$, which corresponds to the solution concept of the "optimal steady state"—a planner entirely focused on maximizing steady state welfare.

3.1 Ramsey problem

We follow the dual approach: the Ramsey planner maximizes utilitarian welfare each period, discounted at δ,

$$\sum_{t=0}^{\infty} \delta^t \mathcal{U}_t \left(\{r_s, w_s\}\right) \tag{20}$$

by choice of the time path of prices $\{r_s, w_s\}_{s=0}^{\infty}$, subject to either of the two implementability conditions in proposition 2.[5] It is simpler to work with the government budget constraint (16) as an implementability condition, and we do this from now on. Intuitively, we can think of $w_s = 1 - \tau_s$ as being controlled directly by the planner, and r_s as being endogenously determined by the government budget constraint.

While (20) assumes a utilitarian objective function, this is not crucial for our results, because our model does not have any permanent types. Indeed, any non-utilitarian welfare function with weight κ_i on the utility of individual i will, in the long run, be proportional to a utilitarian welfare function: $\int_i \kappa_i u(c_{it}, n_{it}) di$ is eventually proportional to $\mathcal{U}_t$ because, absent permanent types, individuals are mixing and eventually all look alike. This logic doesn't apply with permanent types, a case that we briefly explore in section 8.1.

3.2 Ramsey steady state

We define a Ramsey steady state as follows.

Definition 3. A steady-state equilibrium consisting of quantities Y, N, B, C, A, a tax rate τ, and prices r, w is called a *Ramsey steady state* of the economy if there exists a solution $\{r_s, w_s\}_{s=0}^{\infty}$ of the Ramsey problem (a *Ramsey plan*) such that $\mathcal{C}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right) \to C$, $\mathcal{A}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right) \to A$, $\mathcal{N}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right) \to N$, $w_t \to w$, and $r_t \to r$.

This means that a Ramsey steady state always has an associated Ramsey plan which converges to it. The following lemma shows that really all we need is that r_t and w_t converge.

Lemma 1. *Under assumption* **??**, *a steady-state equilibrium with prices r, w is a Ramsey steady state if and only if there is a Ramsey plan $\{r_s, w_s\}_{s=0}^{\infty}$ with $r_t \to r$ and $w_t \to w$.*

This lemma is helpful because it shows that we only need convergence of the two price sequences; convergence of all other equilibrium objects follows.

Before we state our main result, we define two additional objects. The first is a unit-less measure of liquidity in any steady state of the economy, defined as the ratio of available assets for self

[5] This planning problem allows the planner to choose the date-0 return r_0. This has no bearing on any of our results; everything goes through if this is ruled out.

insurance to aggregate after-tax income

$$\ell \equiv \frac{A}{wN}$$

Comparing liquid assets to income is a typical way to assess liquidity in the literature (e.g. Kaplan, Violante and Weidner 2014). It is a natural normalization in our model, since the amount of income risk households are subject to scales with aggregate after-tax income wN. Thus, ℓ effectively scales the available liquid assets by the amount of income risk in the economy.

The second object we define is the effective marginal rate of substitution between a one-time interest rate increase and a one-time wage increase around some steady state,

$$\text{mrs}(\delta) \equiv \frac{\epsilon^{U,w}(\delta)}{\epsilon^{U,r}(\delta)} \tag{21}$$

The numerator as well as the denominator are discounted elasticities. When $\delta = \beta$, we show in appendix B.4 that $\text{mrs}(\beta)$ can be rewritten as

$$\text{mrs}(\beta) = \frac{1}{\ell} \cdot \frac{\int u_c(c_{it}, n_{it}) \frac{e_{it} n_{it}}{N} di}{\int u_c(c_{it}, n_{it}) \frac{a_{it-1}}{A} di} \tag{22}$$

The formula follows directly from the envelope theorem. The numerator is the marginal utility of the average labor income recipient; the denominator is the marginal utility of the average asset income recipient. Since labor income recipients are typically poorer in the models we study, we expect $\ell \cdot \text{mrs}(\beta)$ to be greater than one.

3.3 Ramsey steady state condition

A useful way to start thinking about the Ramsey steady state is to derive first-order conditions of the Ramsey problem. For example, denoting by λ_t the date-t current-value Lagrange multiplier of the government budget constraint, we can write the first order condition with respect to r_s, for some period $s \geq 0$, as

$$\sum_{h=-s}^{\infty} \delta^h \frac{\partial \mathcal{U}_{s+h}}{\partial r_s} + \sum_{h=-s}^{\infty} \delta^h \lambda_{s+h} \left(\frac{\partial \mathcal{A}_{s+h}}{\partial r_s} + (1 - w_{s+h}) \frac{\partial \mathcal{N}_{s+h}}{\partial r_s} - (1 + r_{s+h}) \frac{\partial \mathcal{A}_{s+h-1}}{\partial r_s} \right) - \lambda_s \mathcal{A}_{s-1} = 0 \tag{23}$$

Many terms in this expression look similar to the discounted elasticities we introduced in section 2.4. For instance, as one takes the limit $s \to \infty$, the first term immediately converges to $\epsilon^{U,r}(\delta)$. If the multiplier λ_t were to be constant, or converging to a constant, the three terms in parentheses would also all converge to discounted elasticities as $s \to \infty$. A similar first order condition can be written down for the wage w_s at some period s.

We next present a series of necessary conditions for a pair of prices (r, w) to be part of a Ramsey steady state. The main difference between the conditions will be assumptions made on the Lagrange multipliers λ_t of the associated Ramsey plan.

Converging multiplier. We begin with the case where the Lagrange multiplier converges to some nonzero constant.

Proposition 3. *If a pair of prices (r, w) is part of a Ramsey steady state of a Ramsey plan with converging Lagrange multipliers λ_t, then two conditions have to hold:*

1. *The steady state government budget constraint,*

$$G + r\mathcal{A}^{ss}(r, w) = (1 - w)\mathcal{N}^{ss}(r, w) \tag{24}$$

2. *Either the following RSS optimality condition,*

$$\underbrace{(1 - \delta(1 + r))\,\ell\left(mrs\,\epsilon^{A,r} + \epsilon^{A,\tau}\right)}_{\text{liquidity benefit}} \underbrace{- \frac{1 - w}{w}\left(-\epsilon^{N,\tau} - mrs\,\epsilon^{N,r}\right)}_{\text{labor supply}} \underbrace{- (\ell\, mrs - 1)}_{\text{redistribution}} = 0 \tag{25}$$

where we omitted δ as an argument of the elasticities and the mrs, or the unconstrained optimality conditions $\epsilon^{U,r} = \epsilon^{U,w} = 0$.

Proposition 3 gives us two simple conditions to check whether (r, w) is indeed consistent with an RSS. The first is the government budget constraint. The second is essentially a first-order condition (FOC), (25); we allow for the case of unconstrained optimality $\epsilon^{U,r} = \epsilon^{U,w} = 0$ for completeness, though mathematically this can never be satisfied if $\delta = \beta$ due to the argument underlying (22) (see appendix B.4), and practically is never satisfied in any of our simulations for any other δ either.

The key thought experiment underlying the FOC (25) is a shift towards a higher interest rate r at the RSS together with higher labor income taxation τ (or, equivalently, a lower wage w). We imagine the planner contemplating this shift, balancing the r and the w shift so as to keep utility (20) unchanged. The three terms in (25) consist of three forces whose sum tells us whether the planner can gain resources from undertaking such a compensated variation. If there are no resources to gain, the sum of the three terms nets out to zero. This has to be satisfied at any RSS. We next go over each of the three terms in turn.

The *liquidity benefit* term captures the idea that the planner can raise the present value of resources by issuing more debt when there is a gap between the social discount factor δ and the interest rate. An additional unit of debt issued brings the planner an additional resource, but requires payment of $1 + r$ resources in the future. In present value terms, this gives a net resource gain of $1 - \delta(1 + r)$ per unit of debt issued. The expression $\text{mrs}\,\epsilon^{A,r} + \epsilon^{A,\tau}$ can be interpreted as the net increase in debt—or, equivalently, assets held by households—as a result of the compensated increase in interest rates and labor income taxes. We find $\text{mrs}\,\epsilon^{A,r} + \epsilon^{A,\tau}$ to be positive in all our parameterizations below. Observe that, since $\beta(1 + r) < 1$ in any steady state, the liquidity benefit takes a positive sign whenever the planner uses the private discount factor, $\delta = \beta$, but it can be negative when the planner uses a relatively high discount factor, such as in the optimal steady state case of $\delta \to 1$.

The second term, labeled *labor supply*, captures any disincentive effects on labor supply of greater labor income taxation, via $-\epsilon^{N,\tau}$, and interest rates, via $-\text{mrs}\,\epsilon^{N,r}$. One would intuit that this term is negative, indicating lower labor supply. However, as already anticipated in section 2.4, this will not always be the case.

The final term is a *redistribution* term, which demonstrates that the combined increase in interest rates and labor income taxes has distributional consequences. The planner will, in general, have to use more than one unit of resources in terms of higher interest payments to compensate households for a one unit reduction in after-tax wages, because asset income earners have, on average, lower marginal utility. This is especially clear in the case $\delta = \beta$, where (22) shows that $\ell\,\text{mrs}$ is simply a ratio of weighted marginal utilities.

Our next result shows that (25) can be simplified further if preferences are of the log-separable form (11).

Proposition 4. *Assume $u(c,n)$ is log-separable, as in (11). Then, condition (25) in proposition 3 is equivalent to*

$$(1-\delta(1+r))\,\ell\epsilon^{A,r}(\delta) + \frac{1-w}{w}\epsilon^{N,r}(\delta) + (r\ell+1)\,\epsilon^{U,r}(\delta) - \ell = 0 \tag{26}$$

Surprisingly, only interest rate elasticities appear in (26). The reason behind this result goes back to a specific kind of symmetry between r and w derivatives in a log-separable model, which is explained in detail in Auclert et al. (2024a).

Diverging multiplier. Next, we consider the case where the Lagrange multiplier λ_t diverges to infinity. The results above are inapplicable in that case. Instead we have the following result.

Proposition 5. *If a pair of prices (r,w) is part of a Ramsey steady state of a Ramsey plan with diverging Lagrange multipliers, $\lambda_t \to \infty$ with $\lambda_t/\lambda_{t-1} \to \eta \in [1,\delta^{-1}]$, then the government budget constraint (24) as well as two optimality conditions*

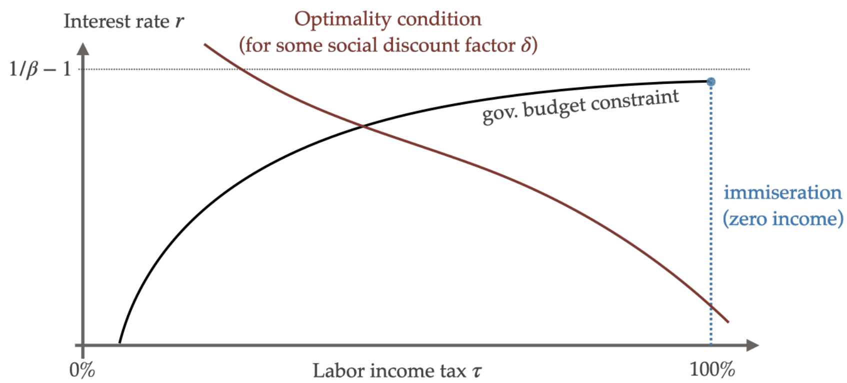
$$(1-\delta\eta(1+r))\,\ell\epsilon^{A,\tau}(\delta\eta) - \frac{1-w}{w}\left(-\epsilon^{N,\tau}(\delta\eta)\right) + 1 = 0 \tag{27}$$

$$(1-\delta\eta(1+r))\,\ell\epsilon^{A,r}(\delta\eta) - \frac{1-w}{w}\left(-\epsilon^{N,r}(\delta\eta)\right) - \ell = 0 \tag{28}$$

have to hold.

A diverging Lagrange multiplier implies that the planner eventually perceives its most important role to be to raise available resources, independent of consequences for utility. The two optimality conditions (27) and (28) capture this idea. If (27) holds, the planner cannot raise additional resources by raising labor income taxes further. If (28) the planner cannot raise additional resources by raising interest rates further. Observe that in neither of the two conditions do derivatives of the utility function enter; and the relevant discount factor is effectively no longer δ, but instead $\delta\eta$, adjusted by the growth rate of the multiplier.

Figure 2: Two conditions that determine the Ramsey steady state



It is intuitive that there are three conditions in this case, (24), (27) and (28), as there are three unknowns to pin down, namely r, w, and η.

3.4 Graphical illustration of the optimality condition

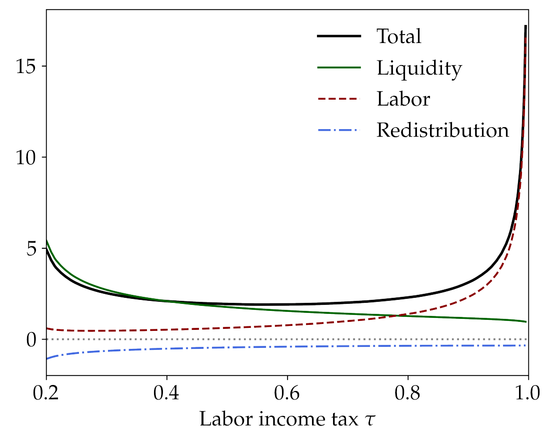
Figure 2 graphically illustrates the determination of the Ramsey steady state in the case of a converging Lagrangian multiplier. We look for a pair of an interest rate r (on the y axis) and a labor income tax rate $\tau = 1 - w$ (on the x axis) such that two conditions hold: *(i)* the government budget constraint (24) (black line) requires that higher interest rates, and thus liquidity provision by the government, needs to be financed by greater labor income taxes; *(ii)* the optimality condition (25) (red line) requires that we optimally resolve the trade-off between interest rates and taxes. The budget constraint in r, τ space typically slopes up, while the optimality condition typically slopes down, though we will encounter some exceptions to this general rule in section 6.[6]

4 Immiseration in Aiyagari models

We now put our optimality conditions into action. In this section, we focus on log-separable preferences (11). We discuss alternative preferences in section 6. Our main result in this section is that if social and private discount factors coincide, $\delta = \beta$, the two lines in figure 2 typically do not cross. Moreover, there is no rate η at which the Lagrange multiplier could diverge that gets the three conditions of proposition 5 to hold. This leads us to conclude that in many standard Aiyagari models, there is no well-defined Ramsey steady state. Instead, we find suggestive evidence that the economy tends to immiseration, with labor income taxes τ approaching 100%.

[6] One aspect of figure 2 is that the highest interest rate consistent with the government budget constraint is strictly below the private discount rate $1/\beta - 1$. In our economy with proportional labor taxation, this is because of positive government spending $G > 0$. Bewley (1983) offers a separate reason in an environment with lump-sum taxes: As taxes and liquidity rise sufficiently, agents become Ricardian and the interest rate ceases to rise.

Figure 3: Net benefit from higher liquidity at Ramsey steady state (RSS)



Note: This figure displays the three terms of the interior Ramsey steady state optimality condition (stated in proposition 4) and their sum under the baseline calibration (see section 4.1 for further details). Households have log-separable preferences and a zero-borrowing constraint, and the planner shares the households' preferences, $\delta = \beta$.

4.1 Baseline calibration

As a starting point, we use the following baseline calibration of the economy. The disutility from labor $v(n)$ in (11) is isoelastic with a Frisch elasticity of 1. The Markov chain for idiosyncratic productivities e_{it} follows an AR(1) income process with annual persistence 0.90 and a cross-sectional standard deviation of log e_{it} equal to 0.92. The initial steady state of the economy is one with debt B equal to 100% of GDP, government spending G equal to 20% of GDP, and an interest rate of 2%. This implies a private discount factor of $\beta = 0.897$ to satisfy asset market clearing; and a tax rate of $\tau = 22\%$ to satisfy the government budget constraint.

Below we provide an extensive sensitivity check for how our results depend on these parameters. We find that for a wide range for standard calibrations, the economy either tends to immiseration, without a Ramsey steady state, or to a Ramsey steady state with very high tax rates, often above 90%.

4.2 In search of a Ramsey steady state

We begin our search for a Ramsey steady state by evaluating the two conditions in proposition 3 in the case where the planner shares the households' preferences, $\delta = \beta$. We do so by varying the potential RSS labor income tax τ, then solving for the associated interest rate r that satisfies the government budget constraint (24), and finally by evaluating the optimality condition (25) for that pair of τ and r. As before, we note that w is simply equal to $1 - \tau$. Graphically, we can think of the approach as walking from the bottom left to the top right corner along the black solid line in figure 2, always evaluating the optimality condition (25), seeing whether it ever crosses zero.

Figure 3 plots the three terms on the left hand side of the optimality condition (25) (various colors) as well as the sum of all terms (black, solid). As expected, the liquidity effect is positive. Furthermore, the redistribution is negative. However, the labor supply effect is *positive*, indicating that greater labor taxes and interest rates lead to a *positive* response of the present discounted value of labor supply—much in line with our discussion on the anticipatory labor supply effects of future labor income taxes in section 2.4.

The positive labor supply effect prevents the sum of all three effects (black solid) from ever crossing zero. There is no Ramsey steady state with a converging Lagrange multiplier in this economy. In figure D.1 in the appendix, we check the first order conditions with exploding multipliers and also do not find a Ramsey steady state.

4.3 Optimal steady state and other social discount factors

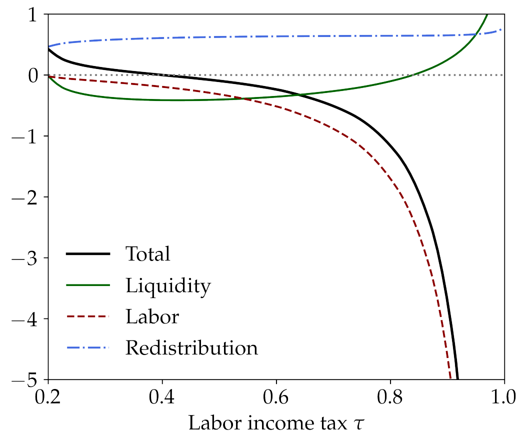
Figure 4 repeats the same calculation for the case of a social discount factor equal to 1, $\delta = 1$. This can be interpreted as a planner that is maximizing steady state welfare only, ignoring the transition it takes to reach the steady state. Note that assuming that planner and household preferences are misaligned makes the three terms in (25) harder to interpret. For example, the first term capturing the liquidity benefit is now negative, as the planner no longer uses a discount factor below $1/(1+r)$, like β, and instead ceases to discount altogether. Additionally, the redistribution term is now positive as greater steady state interest rates no longer simply benefit today's asset rich households, but *any* household, as they all hold some assets eventually.

The labor supply term, however, is still capturing disincentive effects for greater interest rates and labor income taxation. As figure 4 shows, these now go in the intuitive direction and turn negative eventually. This ensures a unique intersection of our optimality condition with zero, and hence a unique optimal steady state (OSS) in this economy.

Similar to our procedures for $\delta = \beta$ and $\delta = 1$, we can also search for a Ramsey steady state for any intermediate social discount factor δ. Figure 5 follows this approach and computes various steady state outcomes, such as consumption, wages, or liquidity, whenever a candidate Ramsey steady state for a given δ was found. The figure shows that such a steady state can be found for δ's very close to β.[7] Interestingly, as δ is moved from 1 all the way to the left, towards β, many quantities and prices approach a corner. For example, consumption converges to zero, GDP converges to government spending G, and labor taxes converge to 100%. This is a first indication as to what might happen in lieu of convergence to a Ramsey steady state when $\delta = \beta$: the economy might tend to long-run immiseration.

[7] There is a small interval of positive measure $[\beta, \tilde{\beta})$ just above β in which social discount factors do not give rise to a candidate Ramsey steady state.

Figure 4: Net benefit from higher liquidity at the optimal steady state (OSS)



Note: This figure displays the three terms of the interior Ramsey steady state optimality condition (stated in proposition 4) and their sum under the baseline calibration (see section 4.1 for further details). The planner maximizes steady-state welfare, $\delta = 1$.

4.4 Immiseration

Our results in figure 5 suggest that the Ramsey plan of a planner that shares the household preferences, i.e. $\delta = \beta$, tends to immiseration: tax rates τ_t rise to 100%, after-tax wages w_t fall to zero. None of our conditions in section 3.3 apply to this situation as there is no well-defined long-run steady state equilibrium here. We next provide a necessary condition that specifically captures immiseration dynamics in the case of log-separable preferences (11).

Proposition 6. *Assume $u(c, n)$ is of the log-separable form, (11). Let $\{r_t, w_t\}$ be an optimal Ramsey plan such that:*

- *w_t falls to zero with asymptotic decay factor $\gamma \in [\delta, 1)$, that is, $\lim_{t\to\infty} w_t/\gamma^t$ exists and is positive.*
- *r_t converges to some constant $r < \gamma/\beta - 1$.*
- *λ_t diverges with asymptotic factor $\eta \in (1, \delta^{-1}]$, that is, $\lim_{t\to\infty} \lambda_t/\eta^t$ exists and is positive.*

Then, the following two conditions have to hold, evaluated at a de-trended steady state with interest rate $1 + \hat{r} = (1 + r)/\gamma$, the same discount factor β, and wage $\hat{w} = 1$:

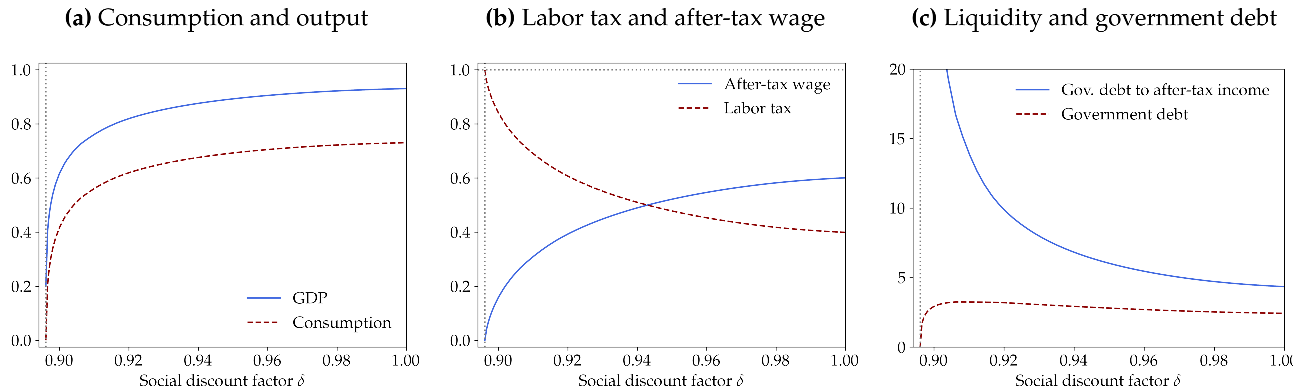
1. *an immiseration-adjusted government budget constraint*

$$G = \mathcal{N}^{ss}(\hat{r}, \hat{w}) \tag{29}$$

2. *an immiseration-adjusted optimality condition*

$$\epsilon^{N,\tau}(\delta\eta) = \epsilon^{N,r}(\delta\eta) = 0 \tag{30}$$

Figure 5: Ramsey steady state as function of social discount factor



Note: Each panel in this figure displays the interior Ramsey steady state values of variables attained as the social discount factor δ is varied from $\delta = \beta$ (grey-dotted vertical line) to $\delta = 1$ under the baseline calibration (see section 4.1 for further details).

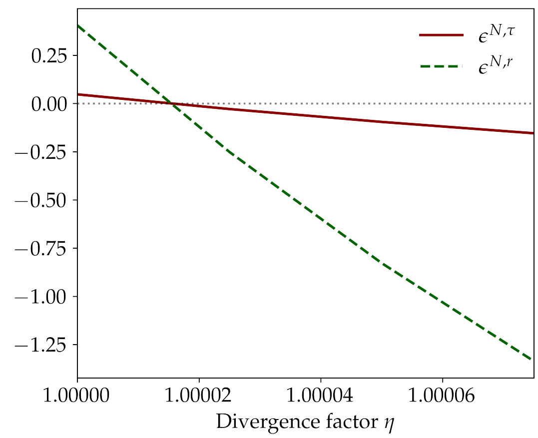
Proposition 6 is another necessary condition, which has to hold if the economy tends to immiseration at an exponential rate. It is based on the idea that a household side whose after-tax wages are shrinking at an exponential rate can be "de-trended". With log-separable preferences, (11), the de-trended household side is one whose labor supply is given by $\mathcal{N}^{ss}(\hat{r}, \hat{w})$, where $1 + \hat{r} = (1 + r) / \gamma$ is a "de-trended interest rate" and the wage $\hat{w}$ is normalized to 1; and whose asset demand is given by $w_t \cdot \mathcal{A}^{ss}(\hat{r}, \hat{w})$. With these relationships, the government budget constraint (24) collapses to (29).

The optimality condition (30) can, at least heuristically, be derived from our optimality conditions with diverging multipliers (27) and (28). For example, the only way (27) can continue to hold as $w \to 0$ is if $\epsilon^{N,\tau}(\delta\eta) = 0$. Similarly, $\epsilon^{N,r}(\delta\eta) = 0$ follows from (28) in this limit. It turns out that with log-separable preferences, $\epsilon^{N,\tau}(\delta\eta) = 0$ precisely holds if and only if $\epsilon^{N,r}(\delta\eta) = 0$, for arbitrary $\delta\eta$. Thus, (30) should be thought of as a single condition, not two independent ones. Given any δ, (30) can then be used to solve for η.

A limitation of proposition 6 is that it does not allow us to back out the rate at which the economy tends to immiseration. Instead, (29) and (30) are two conditions that determine the de-trended interest rate $\hat{r}$ as well as the factor η with which the Lagrange multiplier diverges. To show how this works in our baseline calibration, we solve (29) for our baseline economy, finding $\hat{r} = 11.47\%$. Figure 6 plots $\epsilon^{N,\tau}(\beta\eta)$ and $\epsilon^{N,r}(\beta\eta)$ varying η. As the figure clearly shows, an η just above 1 is sufficient to turn $\epsilon^{N,\tau}(\beta\eta)$ and $\epsilon^{N,r}(\beta\eta)$ to zero.[8]

[8]We have done a large number of robustness checks to ensure that this finding is not driven by any numerical issues. It is easy to get intersections at zero with significantly greater η's by increasing income risk.

Figure 6: Looking for immiseration



Note: This figure displays labor supply elasticities with respect to labor income taxes and interest rates for the optimal Ramsey plan whose Lagrange multipliers diverge at exponential rate η. Parameters are given by the baseline calibration (see section 4.1 for further details).

5 Alternative calibrations

Our immiseration result raises many questions. First among them is the role our baseline calibration plays. In this section, we vary income risk, income inequality, and the Frisch elasticity. Appendix F provides further model variants and extensions, with endogenous government spending, a fixed lump-sum transfer, an open economy setting, and an economy with private borrowing.

5.1 Role of income risk

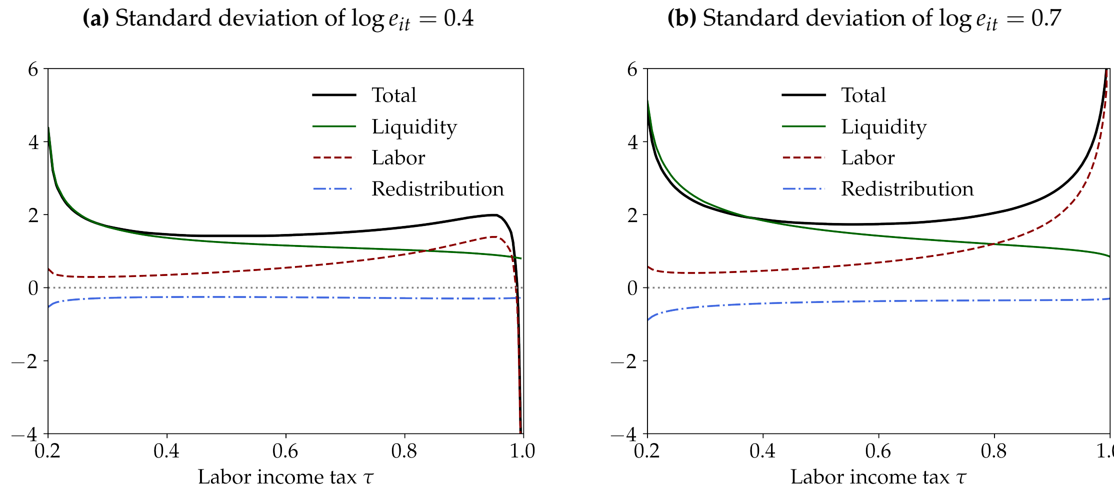
Figure 7 plots the left hand side of the optimality condition (25) for lower income risk. Panel (a) reduces the variance of the innovations to $\log e_{it}$ such that its cross-sectional standard deviation is 0.4 rather than 0.91. This yields a unique candidate Ramsey steady state, with a labor tax rate very close 100%. Panel (b) raises the standard deviation to 0.7; this time, a candidate Ramsey steady state no longer exists.

5.2 Role of income inequality

One reason why there is no candidate Ramsey steady state in figure 3 could be the quantitatively small redistribution cost. In the following, we explore larger redistribution costs using a specification that gives rise to extreme levels of income inequality.

We modify our AR(1) income process in two ways. First, we raise the standard deviation of the innovations to $\log e_{it}$ such that $\log e_{it}$ has a standard deviation of 1.5, far bigger than even the high standard deviation of pre-tax income in the U.S. economy, which is around 0.90. Second, we add an

Figure 7: Net benefit of higher liquidity with different income risk



Note: This figure displays the three terms of the interior Ramsey steady state optimality condition (stated in proposition 4) and their sum for modified calibrations with lower idiosyncratic income risk.

additional state to the Markov chain, which we call a "poverty state". We assume a fixed fraction $\mu \in (0, 1)$ of households is permanently in the poverty state.[9] We also make the extreme assumption that households in the poverty state essentially have a labor productivity of zero, $e_{it} \to 0$.

We show in appendix B.16 that the inclusion of such a poverty state modifies the optimality condition (25) in that mrs(δ) is now given by

$$\text{mrs}^{pov}(\delta) = \frac{\epsilon^{U,w}(\delta) + \frac{\mu}{1-\mu}}{\epsilon^{U,r}(\delta)}$$

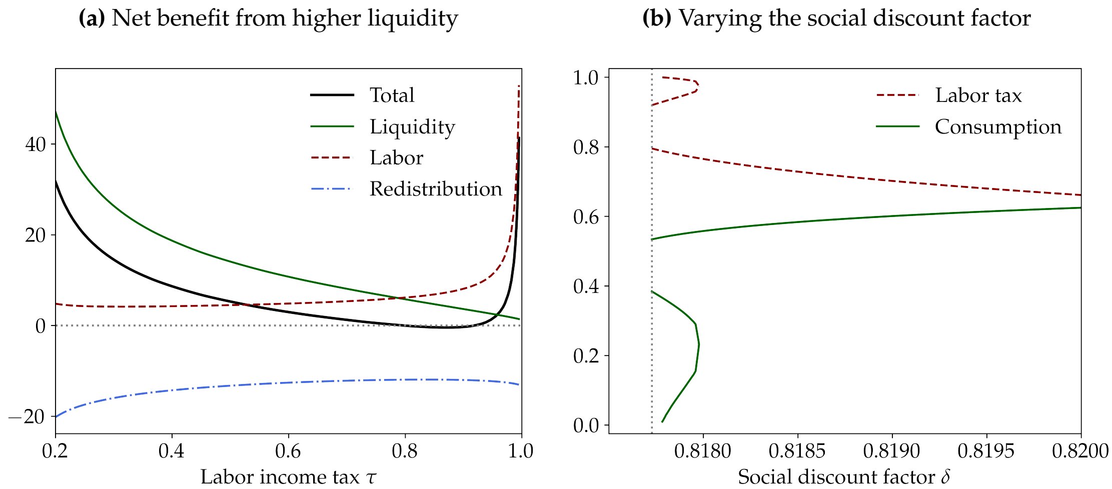
instead of simply $\epsilon^{U,w}(\delta)/\epsilon^{U,r}(\delta)$. This is intuitive, as households in the poverty state see no reason to accumulate assets and thus rely entirely on labor income. They are especially affected by labor income tax increases and do not benefit at all from higher interest rates. All other objects in (25) are independent of μ (including all discounted elasticities and liquidity ℓ).

Figure 8(a) plots all three terms of (25) with our extreme calibration. We see that, indeed, the redistribution term is sufficiently large to pull the sum of all three terms down to zero. Two candidate Ramsey steady states emerge. Both have relatively high labor taxation—one at around 70%, the other at around 95%. Though we cannot check second-order conditions, we speculate that the lower tax steady state is a local maximum, while the higher tax steady state is a local minimum.

Importantly, however, to the right of the high tax steady state, the left hand side of (25)—the

[9]Strictly speaking, this means the Markov chain is no longer recurrent as there are no transitions in or out of the poverty state. We can easily approximate such a Markov chain by assuming households transition into the poverty state with a probability p and leave the poverty state with probability q. The mass of households in the poverty state is then given by $\mu = p/(p+q)$. Assuming that $p, q \to 0$, but fixing the ratio q/p, we can target a specific fraction μ in the poverty state.

Figure 8: Ramsey steady states with extreme income inequality



Note: Panel (a) displays the three terms of the interior Ramsey steady state optimality condition (stated in Proposition 3) and their sum for a modified calibration where 90 percent of households are hand-to-mouth and the remaining households face heightened idiosyncratic income risk, with a cross-sectional standard deviation of $\log e_{it}$ of 1.5. Panel (b) displays the interior Ramsey steady state values of labor taxes and aggregate consumption attained as the social discount factor δ is varied from $\delta = \beta$ (grey-dotted vertical line) to a higher value of δ.

net benefit from further liquidity creation—is positive again. This suggests that, despite having found two candidate Ramsey steady states, immiseration may still be possible. Figure 8(b) follows our approach from figure 5 and computes Ramsey steady states varying the social discount factor δ. We see that, zooming in close to β, there are, in fact three candidate Ramsey steady states: two that converge to those found in figure 8(a), and one that leads to immiseration. This corroborates our suspicion that even extreme inequality does not affect the viability of immiseration as long-run outcome of the Ramsey plan.

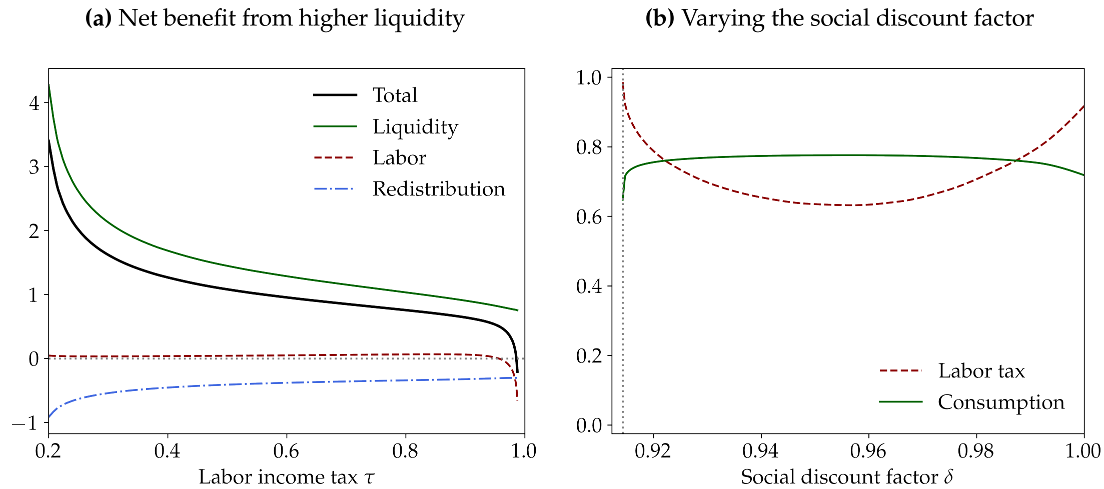
We can also apply the necessary conditions for immiseration from proposition 6. As the economy still has log-separable preferences, and (30) is independent of the mrs, the immiseration conditions are satisfied for the same pair of $\hat{r}$ and η irrespective of the size of the poverty state μ.

5.3 Role of the Frisch elasticity

We next study the role of the Frisch elasticity. Our baseline calibration assumed a Frisch elasticity of 1. Varying the Frisch elasticity in a reasonable range doesn't affect our results qualitatively. Our results do change, however, when the Frisch elasticity is assumed to be very close to zero.

Figure 9(a) plots the net benefit from higher liquidity for an economy with a Frisch elasticity of 0.05. The labor supply margin is now nearly absent for all reasonable levels of labor taxes. For taxes just below 100%, however, we find that the labor supply margin turns negative, implying that in this case, a candidate Ramsey steady state exists, albeit one very close to immiseration, with tax

Figure 9: Ramsey steady states with relatively inelastic labor supply



Note: Panel (a) displays the three terms of the interior Ramsey steady state optimality condition (stated in proposition 3) and their sum for a modified calibration with a Frisch elasticity of 0.05. Panel (b) displays the interior Ramsey steady state values of labor taxes and aggregate consumption attained as the social discount factor δ is varied from $\delta = \beta$ (grey-dotted vertical line) to $\delta = 1$.

rates around 99%. Panel (b) shows that consumption tends to remain constant as we vary δ in this low Frisch elasticity parameterization. Indeed, in the limit of a zero Frisch elasticity, consumption would be exactly constant, equal to the constant level of labor N minus government spending G. Still, we suspect that in this limit, Ramsey steady state tax rates are close to 100%.

6 Alternative preferences

We now start considering classes of preferences other than the log-separable ones used in the previous section. We consider three kinds: general additively separable preferences; Greenwood et al. (1988) preferences without wealth effects on labor supply; and general balanced growth preferences à la King et al. (1988).

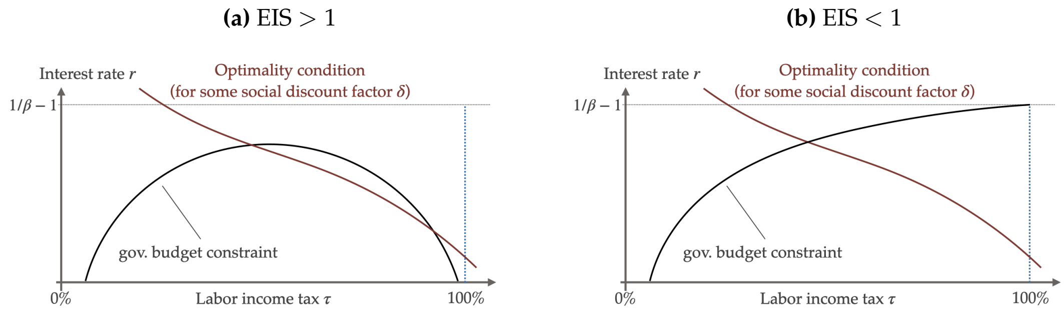
6.1 Additively separable preferences

We begin by considering additively separable preferences,

$$u(c,n) = \frac{c^{1-\sigma}-1}{1-\sigma} - \phi\frac{n^{1+\nu}}{1+\nu} \tag{31}$$

where $\phi > 0$ is a constant and ν is the inverse of the Frisch elasticity of labor supply. For $\sigma \neq 1$, these preferences are no longer compatible with balanced growth. This has a significant effect on

Figure 10: Two RSS conditions with additively separable preferences



the possible long-run behavior of Ramsey plans. To see why, consider the two diagrams in figure 10.

Panel (a) focuses on the case where the EIS $1/\sigma$ lies above 1. It is well known that, with these preferences, the static substitution effect from wage changes is stronger than the static income effect. This implies that the steady state government budget constraint looks very different from the one in figure 2. As the labor tax gets closer to 100%, the substitution effect becomes sufficiently strong that households increasingly work less. This reduces tax revenue and hinders liquidity provision, lowering the equilibrium interest rate again. The government budget constraint thus exhibits a hump-shaped profile.

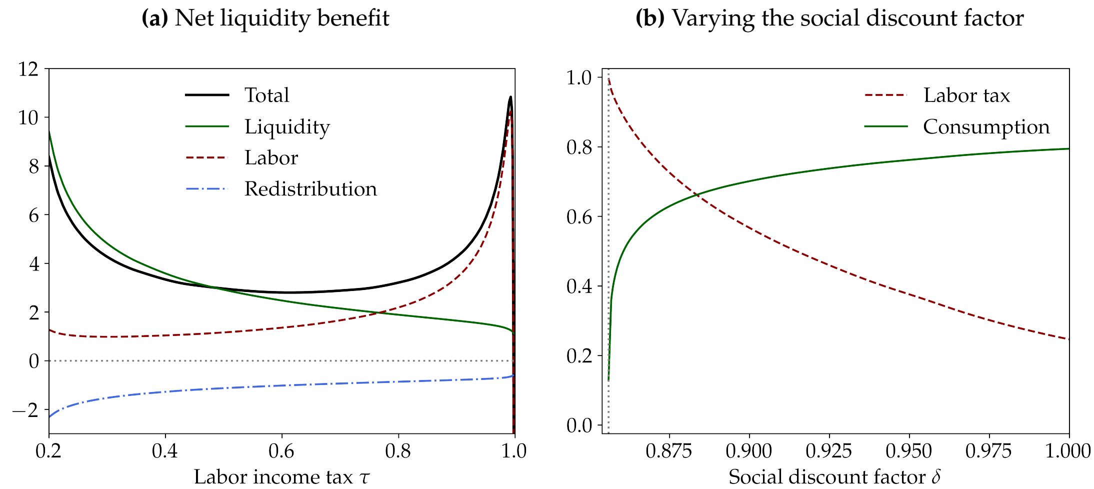
Panel (b) focuses on the case where the EIS $1/\sigma$ lies below 1. Here, we have the opposite. As labor taxes rise, households feel increasingly poor, and start working harder, generating additional tax revenue that allows the government to raise liquidity provision and raise the equilibrium interest rate r.

EIS > 1. We numerically search for candidate Ramsey steady states in the case where the EIS $1/\sigma$ is above 1. Other than the different preferences, the calibration is the same as the one introduced in section 4.1. Due to the hump-shaped profile in figure 10(a), there can be anywhere between zero and two intersections of the optimality curve with the government budget constraint. Zero intersections are more likely when the budget constraint line is lower, as is for example the case with higher levels of government consumption G.

With low government spending G, there are in principle two candidate Ramsey steady states for each social discount factor $\delta \geq \beta$, as illustrated in figure 10(a). For an EIS of 2 in our calibrated model, we found these to be extremely close to 100% labor taxation. Figure D.2 in the appendix instead considers greater government spending $G = 0.31$. This is enough to prevent any intersections between the two curves in figure 10(a). When we vary the social discount factor δ above β, we see that for sufficiently high δ, two candidate Ramsey steady states start emerging.

What happens in the high G calibration when $\delta = \beta$? Evidently, there is no combination of r and w that satisfies the conditions of proposition 3 for a Ramsey steady state that is reached with converging multipliers. It turns out, however, that in this case, there *is* a solution (r, w, η) of the

Figure 11: RSS for additively separable preferences with EIS < 1



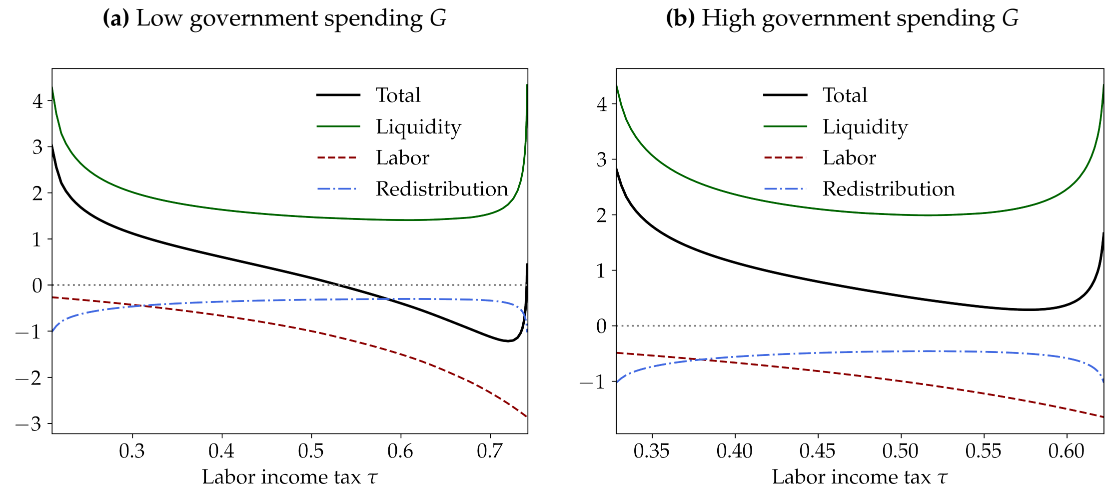
Note: Panel (a) displays the three terms of the interior Ramsey steady state optimality condition (stated in proposition 3) and their sum for a modified calibration with EIS $\sigma^{-1} = 0.5$. Panel (b) displays the interior Ramsey steady state values of labor taxes and aggregate consumption attained as the social discount factor δ is varied from $\delta = \beta$ (grey-dotted vertical line) to $\delta = 1$.

condition with diverging multipliers, per proposition 5.

EIS < 1. With an EIS below 1, matters are conceptually simpler. Figure 11(a) shows that there exists a unique candidate Ramsey steady state, albeit one with labor income taxes incredibly close to 100%, suggesting the economy is going to near-immiseration in this case. Intuitively, with a strong income effect, tax rates do not need to rise all the way to 100% to provide households with an incentive to increase their labor supply.

Taking stock: additively separable preferences. To summarize, the results we obtain with the additively separable preference class are also quite extreme. When the EIS is above 1, one or more Ramsey steady states may exist, all of which with very high tax rates. When the EIS is below 1—which is more consistent with the literature on structural change (Boppart and Krusell 2020)—our results suggest that labor taxes get very close to 100%, but do not reach 100%, as the strong income effect provides a sufficiently strong incentive for households to raise labor supply, avoiding full immiseration.

Figure 12: RSS for GHH preferences



Note: The figures display the three terms of the interior Ramsey steady state optimality condition (stated in proposition 3) and their sum for a modified calibration with GHH preferences. Government spending-to-GDP ratio is $G/Y = 0.25$ in panel (a) and 0.31 in panel (b).

6.2 Greenwood et al. (1988) preferences

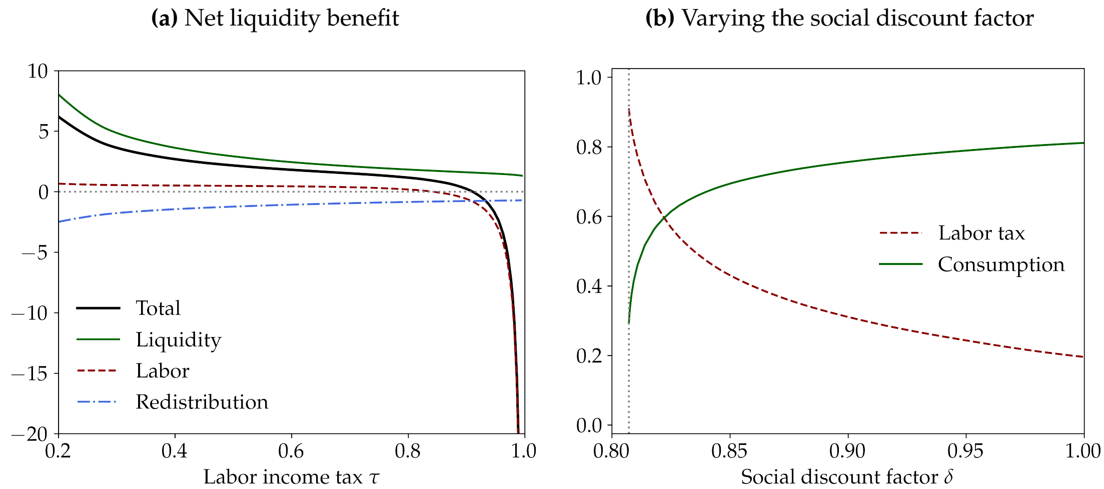
We next move on to study Greenwood et al. (1988) preferences, or GHH preferences for short:

$$u(c,n) = \frac{\left(c - \phi\frac{n^{1+\nu}}{1+\nu}\right)^{1-\sigma} - 1}{1-\sigma}$$

As is well-known, these preferences are designed to feature no income effects on labor supply. Therefore, these preferences are conceptually similar to additively separable preferences (31) with an EIS $1/\sigma > 1$, in that they both feature a dominant substitution effect. This is why the r, τ diagram of a GHH economy looks very similar to that shown in figure 10(a). In particular, depending on the magnitude of government consumption, there can be anywhere between zero and two intersections; and if there are zero intersections, the conditions from proposition 5 that allow for an exploding Lagrange multiplier can be applied.

Using the same calibration as in section 4.1, we show analogous results to additively separable preferences with an EIS $1/\sigma > 1$. With government spending of $G = 0.25$, we find two potential Ramsey steady states (figure 12a). With slightly higher government spending of $G = 0.31$, however, there no longer is a candidate Ramsey steady state with a converging multiplier λ_t, as figure 12b shows. In this case, we can again find a Ramsey steady state, once we apply the conditions from proposition 5 that allow for a diverging multiplier λ_t.

Figure 13: RSS with balanced growth preferences and EIS < 1



Note: Panel a displays the three terms of the interior Ramsey steady state optimality condition (stated in Proposition 3) and their sum for a modified calibration with balanced growth preferences and EIS $\sigma^{-1} = 0.5$. Panels b displays the interior Ramsey steady state values of labor taxes and aggregate consumption attained as the social discount factor δ is varied from $\delta = \beta$ (grey-dotted vertical line) to $\delta = 1$.

6.3 Balanced growth preferences à la King et al. (1988)

For completeness, we briefly consider general balanced-growth compatible preferences. The main case we focus on here is the case of an EIS below 1.[10]

Figure 13 considers, for instance, the case of an EIS of 0.5. Here, we do find a Ramsey steady state, as the contribution of the labor term eventually turns negative. However, the associated tax rate is very high, above 90%, and consumption is only 30% of the calibrated steady state output.

7 Capital and capital taxes

So far, we have entirely focused on an economy with only a single asset for households to invest in—government debt—and only labor in the production function. Next, we allow for capital both in production and as an asset for households to self-insure with. Like Aiyagari (1995), we also allow the planner to raise capital taxes. We will show that the conclusions we derived above qualitatively carry over to this economy. For further extensions that allow for additional tax instruments, such as lump-sum or progressive taxes, see appendix E.

[10] For values of the EIS above one, we have not been able to consistently find a unique stationary household wealth distribution at the initial steady state. We suspect that this is a consequence of the substitutability between hours and consumption induced by $\sigma^{-1} > 1$.

7.1 Environment with capital

Instead of (12), the production function is now

$$Y_t = K_{t-1}^{\alpha} N_t^{1-\alpha} \tag{32}$$

for some $\alpha \in (0,1)$.[11] Capital K_t is chosen one period in advance, and depreciates at rate $\delta_k > 0$. Firm optimization implies that the pre-tax real wage w_t^* and the pre-tax return on capital r_t^* are given by

$$w_t^* = (1-\alpha)\left(\frac{K_{t-1}}{N_t}\right)^{\alpha} \qquad \text{and} \qquad r_t^* = \alpha\left(\frac{N_t}{K_{t-1}}\right)^{1-\alpha} - \delta_k$$

We allow for a capital income tax τ_t^k, such that the after-tax return on capital r_t is given by

$$r_t = r_t^* \left(1 - \tau_t^k\right) \tag{33}$$

The after-tax wage is now equal to

$$w_t = w_t^* \left(1 - \tau_t\right) \tag{34}$$

Finally the government budget constraint now accounts for capital tax revenue

$$G + (1 + r_t) B_{t-1} = B_t + \tau_t w_t^* N_t + \tau_t^k r_t^* K_{t-1} \tag{35}$$

We define equilibrium analogously to definition 2.

Definition 4. A *competitive equilibrium* in our economy with capital is a collection of quantities $\{Y_t, N_t, B_t, C_t, A_t, K_t\}_{t=0}^{\infty}$, tax rates$\{\tau_t, \tau_t^k\}_{t=0}^{\infty}$, and prices $\{r_t, w_t\}_{t=0}^{\infty}$, such that:

1. The after-tax wage is given by (34), and the after-tax return on capital is given by (33)
2. Households optimize given prices: aggregate consumption C_t is given by (6), aggregate assets A_t are given by (4), and aggregate effective labor supply N_t is given by (5).
3. Output is given by (32).
4. The government budget constraint (35) holds.
5. The asset market clears, $A_t = B_t + K_t$, and the goods market clears, $C_t + K_t - (1-\delta_k) K_{t-1} + G = Y_t$.

A competitive equilibrium is a *steady state equilibrium* if all quantities, tax rates, and prices are constant.

[11] Our results in this section can be generalized to the case of a general production function with constant returns to scale.

Just as in section 2, we can establish two kinds of implementability conditions; one based on the government budget constraint, the other based on the resource constraint.

Proposition 7 (Implementability with capital)**.** $\{r_t, w_t, K_t\}_{t=0}^{\infty}$ *are part of a competitive equilibrium (in the economy with capital) if and only if one of the following two conditions holds: Either*

$$\mathcal{C}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right) + K_t - (1-\delta_k) K_{t-1} + G = K_{t-1}^{\alpha} \mathcal{N}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right)^{1-\alpha} \tag{36}$$

or

$$\begin{aligned} G + K_t - (1-\delta_k) K_{t-1} + (1+r_t) \mathcal{A}_{t-1}\left(\{r_s, w_s\}_{s=0}^{\infty}\right) \\ = \mathcal{A}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right) + K_{t-1}^{\alpha} \mathcal{N}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right)^{1-\alpha} - w_t \mathcal{N}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right) \end{aligned} \tag{37}$$

Different from our earlier implementability conditions in proposition 2, capital now enters the conditions. Aside from that, the conditions are similar to those before. (36) is the goods market clearing condition, now including capital as factor of production on the right hand side, and as investment on the left hand side. (37) can be best understood as the joint budget constraint of the government and the production sector of the economy. On the left hand side, these sectors together spend G, invest $K_t - (1-\delta_k) K_{t-1}$, and pay $(1+r_t) \mathcal{A}_{t-1}\left(\{r_s, w_s\}_{s=0}^{\infty}\right)$ to savers, either in terms of the return on government bonds or the return on capital. On the right hand side, these sectors obtain $\mathcal{A}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right)$ in new saving, $K_{t-1}^{\alpha} \mathcal{N}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right)^{1-\alpha}$ in production revenue, and pay $w_t \mathcal{N}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right)$ to workers.

7.2 Ramsey steady state with capital

The Ramsey planner in the economy with capital still maximizes (20), only now subject to one of the above implementability conditions; for concreteness, we work with (37). The Ramsey planner chooses the sequences $\{r_s, w_s, K_s\}_{s=0}^{\infty}$. We follow essentially the same definition of a Ramsey steady state as in section 3:

Definition 5. In the economy with capital, a steady-state equilibrium consisting of quantities Y, N, B, C, A, K, tax rates τ, τ^k, and prices r, w is called a *Ramsey steady state* of the economy if there exists a solution $\{r_s, w_s, K_s\}_{s=0}^{\infty}$ of the Ramsey problem (a *Ramsey plan*) such that $\mathcal{C}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right) \to C$, $\mathcal{A}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right) \to A$, $\mathcal{N}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right) \to N$, $w_t \to w$, $K_t \to K$ and $r_t \to r$.

And just like before, it suffices to establish convergence of r_t, w_t, and K_t.

Lemma 2. *Under assumption* **??**, *a steady-state equilibrium with prices r, w and capital K is a Ramsey steady state if and only if there is a Ramsey plan $\{r_s, w_s, K_s\}_{s=0}^{\infty}$ with $r_t \to r$, $w_t \to w$, and $K_t \to K$.*

The main difference relative to our analysis in section 3 is that here, the planner has an additional choice variable, namely the path of capital $\{K_t\}_{t=0}^{\infty}$. This leads to a corresponding additional first-order condition, which reads

$$\delta \lambda_{t+1} \left(\alpha \left(\frac{\mathcal{N}_t \left(\{r_s, w_s\}_{s=0}^{\infty} \right)}{K_{t-1}} \right)^{1-\alpha} + 1 - \delta_k \right) = \lambda_t \tag{38}$$

where λ_t is the multiplier on the implementability condition.

Converging Lagrange multiplier. We next derive two kinds of optimality conditions for the economy with capital. As before, one will assume a convergent path of the Lagrange multiplier λ_t, the other a diverging path.

Proposition 8. *If (r, w, K) is part of a Ramsey steady state of a Ramsey plan with converging Lagrange multipliers λ_t, then three conditions have to hold:*

1. *Steady state implementability:*

$$G + r\mathcal{A}(r, w) = K^{\alpha} \mathcal{N}^{ss}(r, w)^{1-\alpha} - \delta_k K - w \mathcal{N}^{ss}(r, w) \tag{39}$$

2. *RSS optimality:*

$$(1 - \delta(1+r)) \ell \left(mrs\, \epsilon^{A,r} - \epsilon^{A,w} \right) - \frac{w^* - w}{w} \left(\epsilon^{N,w} - mrs\, \epsilon^{N,r} \right) - (\ell\, mrs - 1) = 0 \tag{40}$$

where we omitted δ as an argument of elasticities and the mrs, and $w^* = (1-\alpha) \left(\frac{K}{\mathcal{N}^{ss}(r,w)} \right)^{\alpha}$.

3. *Modified golden rule:*

$$\alpha \left(\frac{\mathcal{N}^{ss}(r, w)}{K} \right)^{1-\alpha} = \delta^{-1} - 1 + \delta_k \tag{41}$$

These conditions are similar to those in proposition 3. The optimality condition (40) is nearly identical. The key new condition is the modified golden rule (41), which follows directly from the first order condition for capital (38) and the assumption of a converging λ_t. With (41), we can solve directly for the real wage at any Ramsey steady state,

$$w^* = (1-\alpha) K^{\alpha} N^{-\alpha} = (1-\alpha) \left(\frac{\alpha}{\delta^{-1} - 1 + \delta_k} \right)^{\frac{\alpha}{1-\alpha}} \tag{42}$$

which is a constant and independent of the values of r, w, and K. We can use this to rewrite the RSS conditions as follows.

Corollary 1. *If r, w is part of a Ramsey steady state of a Ramsey plan with converging Lagrange multipliers λ_t, then two conditions have to hold:*

$$G + r\mathcal{A}(r, w) = \left(\frac{w^*}{1-\alpha} - \left(\frac{w^*}{1-\alpha} \right)^{1/\alpha} \delta_k - w \right) \mathcal{N}^{ss}(r, w) \tag{43}$$

and (40).

This shows that the structure of the optimality conditions barely changed relative to section 3. As the corollary shows, we can once more boil it down to two conditions in two unknowns, r, w. K is then implied by the modified golden rule (41).

These conditions can be used to back out the steady state tax rates. Given w^* in (42) and w, τ can be computed from (34). The pre-tax return on capital is given by $r^* = \delta^{-1} - 1$, and τ^k can then be computed from (33).

Diverging Lagrange multiplier. When the Lagrange multiplier diverges, proposition 8 no longer applies. We instead have the following result.

Proposition 9. *If a pair of prices (r, w) is part of a Ramsey steady state of a Ramsey plan with exponentially diverging Lagrange multipliers, $\lambda_t / \lambda_{t-1} \to \eta \in [1, \delta^{-1}]$, then the implementability condition (43) as well as two optimality conditions*

$$- (1 - \delta\eta (1 + r)) \, \ell \epsilon^{A,w}(\delta\eta) - \frac{w^* - w}{w} \epsilon^{N,w}(\delta\eta) + 1 = 0 \tag{44}$$

$$(1 - \delta\eta (1 + r)) \, \ell \epsilon^{A,r}(\delta\eta) - \frac{w^* - w}{w} \left(-\epsilon^{N,r}(\delta\eta) \right) - 1 = 0 \tag{45}$$

have to hold. The real wage is given by

$$w^* = (1 - \alpha) \left(\frac{\alpha}{\delta^{-1}\eta^{-1} - 1 + \delta_k} \right)^{\frac{\alpha}{1-\alpha}} \tag{46}$$

and the modified golden rule (41) *does not hold.*

With an exponentially diverging Lagrange multiplier λ_t, the modified golden rule fails. This follows directly from (38). Different from (41), the marginal product of capital is now given by

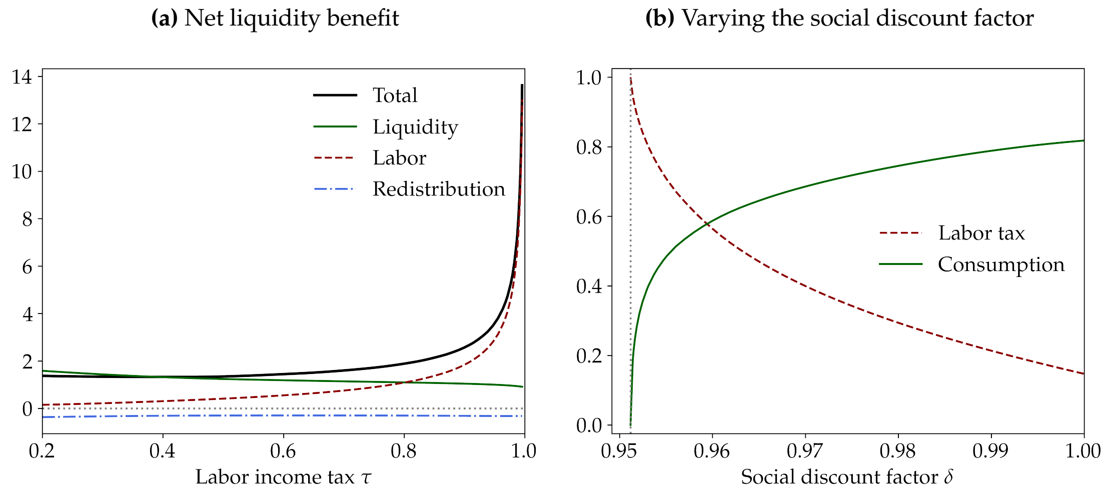
$$\alpha \left(\frac{\mathcal{N}^{ss}(r, w)}{K} \right)^{1-\alpha} = \delta^{-1}\eta^{-1} - 1 + \delta_k$$

and thus is reduced by the growth factor $\eta > 1$ of λ_t.

7.3 Immiseration in the economy with capital

We next investigate whether there exists a Ramsey steady state with converging or diverging multipliers for our baseline calibration. We keep the same calibration targets as in section 4.1. We add two more, namely a capital-output ratio of $K/Y = 3$, a depreciation rate of $\delta_k = 0.05$, and a capital share of $\alpha = 0.3$. This implies a pre-tax return on capital of $r^* = 0.05$ and an initial capital tax of $\tau^k = 60\%$.

Figure 14: Net benefit from higher liquidity in the economy with capital



Note: Panel a displays the three terms of the interior Ramsey steady state optimality condition (stated in Proposition 3) and their sum for a modified calibration including capital, where the capital-output ratio $K/Y = 3$, depreciation rate $\delta_k = 0.05$, and capital share $\alpha = 0.3$. Panel b displays the interior Ramsey steady state values of labor taxes and aggregate consumption attained as the social discount factor δ is varied from $\delta = \beta$ (grey-dotted vertical line) to $\delta = 1$.

We follow the same strategy to find the RSS as in earlier sections of this paper. We vary the labor income tax τ, determining $w = w^*(1-\tau)$ as well as a unique interest rate r that is consistent with (43). For this combination of r, w we then evaluate the left hand side of the first-order condition (40)—the net benefit of higher liquidity. Figure 14 shows that, just like figure 3, the net benefit of greater liquidity is positive throughout. We can similarly show that conditions in proposition 9 do not yield a candidate Ramsey steady state here. Intuitively, everything points to immiseration.

To evaluate immiseration more formally, we next introduce a generalized version of proposition 6.

Proposition 10. *Assume $u(c,n)$ is log-separable, as in (11). Let $\{r_t, w_t, K_t\}$ be an optimal Ramsey plan such that:*

- *w_t falls to zero with asymptotic decay factor $\gamma \in [\delta, 1)$, that is, $\lim_{t\to\infty} w_t/\gamma^t$ exists and is positive.*
- *r_t converges to some constant $r < \gamma/\beta - 1$.*
- *λ_t diverges at with asymptotic factor $\eta \in (1, \delta^{-1}]$, that is, $\lim_{t\to\infty} \lambda_t/\eta^t$ exists and is positive.*

Then, the following two conditions have to hold, evaluated at a de-trended steady state with interest rate $1+\hat{r} = (1+r)/\gamma$, same discount factor β, and wage $\hat{w} = 1$:

1. *an immiseration-adjusted implementability condition*

$$G = \left(\frac{w^*}{1-\alpha} - \left(\frac{w^*}{1-\alpha} \right)^{1/\alpha} \delta_k \right) \mathcal{N}^{ss}(\hat{r}, \hat{w}) \tag{47}$$

2. *an immiseration-adjusted optimality condition*

$$\epsilon^{N,\tau}(\delta\eta) = \epsilon^{N,r}(\delta\eta) = 0 \tag{48}$$

The modified golden rule fails and the pre-tax wage is given by

$$w^* = (1-\alpha)\left(\frac{\alpha}{(\delta\eta)^{-1} - 1 + \delta_k}\right)^{\frac{\alpha}{1-\alpha}} \tag{49}$$

Like before, we can evaluate condition (48) for each η, using the $\hat{r}$ implied by (47). Figure D.4 in the appendix implements this strategy for our baseline calibration. We see that there is a unique intersection, suggesting that, indeed, immiseration is still the outcome in the economy with capital.

8 Alternative household sides

So far, we have focused entirely on a household side in which households draw idiosyncratic productivity shocks from a single stationary Markov chain. In this section, we consider several alternative household sides which are also interesting to study with our approach.

8.1 Permanent types

Imagine that there is a finite set of permanent household types $\mathcal{K}$, where each type $k \in \mathcal{K}$ makes up a fraction μ_k of all households and has a Pareto weight ω_k in the planner's objective function. Denoting the sequence-space functions of each type k by a superscript, e.g. $\mathcal{N}^{(k)}$ for the average labor supply among all type-k households, aggregate behavior of all households is then simply described by the weighted average,

$$\mathcal{N}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right) \equiv \sum_{k\in\mathcal{K}} \mu_k \mathcal{N}_t^{(k)}\left(\{r_s, w_s\}_{s=0}^{\infty}\right) \qquad \text{and} \qquad \mathcal{A}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right) \equiv \sum_{k\in\mathcal{K}} \mu_k \mathcal{A}_t^{(k)}\left(\{r_s, w_s\}_{s=0}^{\infty}\right)$$

and similarly for $\mathcal{C}_t$. Aggregate utility is equal to

$$\mathcal{U}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right) = \sum_{k\in\mathcal{K}} \omega_k \mathcal{U}_t^{(k)}\left(\{r_s, w_s\}_{s=0}^{\infty}\right)$$

With these functions defined, everything else in our approach carries over to this economy with permanent types.

8.2 Bonds in utility

A tractable household side that has been used to approximate heterogeneous-agent models is a model with bonds-in-utility (BU) households (e.g. Michaillat and Saez 2021, Auclert et al. 2024a, Angeletos, Collard and Dellas 2016). In this model, there is a single representative household with preferences

$$\sum_{t=0}^{\infty} \beta^t u(c_t, n_t, a_t)$$

and budget constraint

$$c_t + a_t = (1 + r_t)\, a_{t-1} + w_t n_t \tag{50}$$

Importantly, this household model allows for wealth a_t to enter the per-period utility function directly. Even though this is a model without a complicated wealth distribution, we can still capture the partial equilibrium solution of this utility maximization problem in terms of sequence-space functions $\mathcal{C}_t, \mathcal{A}_t, \mathcal{N}_t$, and $\mathcal{U}_t$.

To show how we can analyze this tractable model using our approach, we assume u takes a standard additively separable form

$$u(c, n, a) = \log c_t - \phi \frac{n^{1+\nu}}{1+\nu} + \chi \log a_t \tag{51}$$

This is the natural BU counterpart of the log-separable preferences (11). In order to study the asymptotic behavior of the Ramsey plan for an arbitrary social discount factor δ in this economy, we have to find expressions for the discounted elasticities of this model around a steady state with given r, w. To get at the elasticities, we write down all the first order conditions of the model, linearize them, and derive expressions for the discounted elasticities from them. We go over all steps in detail in appendix G.1 and only give an example here for the Euler equation of the model.

The linearized Euler equation can be written as

$$-d\log c_t = \beta\,(1+r)\,(d\log\,(1+r_{t+1}) - d\log c_{t+1}) + (1 - \beta\,(1+r))\,(-d\log a_t)$$

Differentiating both sides w.r.t. an interest rate change dr_s at some other date s, we find

$$-\frac{d\log c_t}{dr_s} = \beta\,(1+r)\left(1_{\{s=t+1\}}\frac{1}{1+r} - \frac{d\log c_{t+1}}{dr_s}\right) + (1-\beta\,(1+r))\left(-\frac{d\log a_t}{dr_s}\right)$$

Multiplying this condition by δ^{t-s} on both sides, summing across t and taking the limit $s \to \infty$ yields an expression purely in terms of discounted elasticities,

$$\epsilon^{C,r} = \beta\,(1+r)\left(\delta^{-1}\epsilon^{C,r} - \frac{1}{\delta\,(1+r)}\right) + (1-\beta\,(1+r))\,\epsilon^{A,r}$$

Following this strategy for all first order conditions, as well as the budget constraint (50) and the utility function (51), we can find explicit expressions for the discounted elasticities of N, A, and U w.r.t. r and w (see appendix G.1). Once the elasticities are computed, we can evaluate the first-order optimality condition (25) for Ramsey steady states with a converging Lagrange multiplier.

The optimality condition takes an especially insightful form when the Frisch elasticity is infinite, or $\nu = 0$. In that case, the optimality condition can be written as

$$\frac{\tau}{1-\tau} = \frac{1-\beta(1+r)}{\delta-\beta}\beta \cdot \frac{\chi\left(1-\beta^{-1}\delta\right)+1-\beta(1+r)}{\chi(1-\beta)+1-\beta(1+r)} \tag{52}$$

This describes a relationship between the asymptotic labor tax τ and the interest rate r for each social discount factor δ. The highest steady state interest rate feasible in this economy is

$$1+r^{im} = \frac{1}{\beta}\frac{1+\chi-\phi G}{1+\beta^{-1}\chi-\phi G}$$

where we require that $\phi G < 1$. At interest rate r^{im}, the household works exactly enough to produce G; this is the immiseration interest rate. Since $\beta\left(1+r^{im}\right) < 1$, and thus $\beta(1+r) < 1$ for any feasible steady state interest rate r, it follows directly from (52) that as $\delta \searrow \beta$, the optimal labor tax τ approaches 1. This provides us with a closed form example for how immiseration can also happen in tractable models. In appendix G.1, we show that this conclusion is unchanged if the inverse Frisch elasticity ν is positive.[12]

While there are similarities between our BU analysis here and the one in the full heterogeneous-agent model in section 4, there are also differences. For example, the BU model with preferences (51) has a well-defined Ramsey steady state for any δ arbitrarily close to β. This is not the case in the full heterogeneous-agent economy, where there is a non-zero measure interval of social discount factors above β for which there is no Ramsey steady state.

8.3 Alternating income states

Next, we consider the economy studied in Woodford (1990) in which households face a deterministic sequence of productivities that alternates between 2 ("employed") and 0 ("unemployed"). In appendix G.2, we derive the elasticities for this model when the utility function is log-separable, as in (11). We find that output is constant, $Y_t = \overline{Y}$, in the model and that the elasticities are given by

$$\epsilon^{N,r} = \epsilon^{N,w} = \epsilon^{A,r} = 0 \qquad \epsilon^{A,w} = 1 \qquad \text{mrs} = (1+r)(1+\delta) \qquad \ell = \frac{\beta}{1+\beta}$$

Substituting these expressions into (25), we find $\beta(1+r) = 1$. In other words, the optimality condition in figure 2 is a horizontal line at $\beta^{-1} - 1$. Along this line, households save sufficient funds to avoid a binding borrowing constraint.

[12] This conclusion does not necessarily follow if preferences (51) are not consistent with balanced growth. For example, Angeletos et al. (2016) work with a setup that has a linear utility over consumption, more similar to the GHH preferences analyzed in section 6.2. This is why Angeletos et al. (2016) find two interior Ramsey steady states, just like we did in section 6.2 with sufficiently low government spending.

The government budget constraint can be shown to simplify to

$$\tau = \frac{\frac{G}{\overline{Y}}\left(1+\beta\right)+\beta r}{1+\beta\left(1+r\right)}$$

where $G < \overline{Y}$ to ensure feasibility. At $\beta\left(1+r\right)=1$,

$$\tau = 1-\left(1+\beta\right)\frac{G/\overline{Y}-1}{2}$$

which is strictly below 100%. There is no immiseration in the Woodford economy. Instead, the planner successfully satiates the economy with liquidity. Such satiation is not achievable in our economy with idiosyncratic risk presented in section 2.

8.4 Overlapping generations

We finally discuss the effect of overlapping generations (OLG) on our results. Consider our model of section 2 and assume households die at some rate $\zeta > 0$. Upon death, they are immediately replaced by an offspring that inherits all wealth and starts from the productivity state the parent would have been in absent death. We assume that the offspring's utility enters the parental utility function with some weight $\vartheta \in [0,1]$. The planner's utility function is assumed to include all generations' lifetime utilities, where a generation born at date t enters with weight β^t.

It is straightforward to see that this setup exactly corresponds to one in which households use the private discount factor $\beta\left(1-\left(1-\vartheta\right)\zeta\right)$, but the planner uses a social discount factor of β. When households are perfectly altruistic, $\vartheta = 1$, the discount factors are aligned and our section 4 results with equal social and private discount factors carry over to this economy with overlapping generations. When, instead, $\vartheta < 1$, households put a utility weight on future generations that is smaller than that of the planner, as in Farhi and Werning (2010). In that case, the overlapping generations economy is equivalent to a section 2 economy where the social discount factor lies above the private one.[13]

This reasoning implies that imperfectly altruistic overlapping generations are likely to make our immiseration results in section 4 less likely to occur. If we assume that δ lies 1.3% above β, where 1.3% is equal to the average mortality risk corresponding to an average life expectancy of about 77 years, then a Ramsey steady state with a labor tax of 73% emerges in our baseline economy (figure 5) and of 46% in the economy with capital (figure 14).

[13] In OLG models without constant mortality risk, such as in Conesa et al. (2009), there is a difference between discounting *flow utility* $\mathcal{U}_t$ with a discount factor above β, and discounting future *lifetime utility* across generations with a discount factor above β, because the latter preserves β as relevant discount factor within each generation. We suspect that our approach can be applied to such models by defining $\mathcal{U}_t$ as lifetime utility of the generation born at date t. We leave a full exploration of this for future work.

9 Relationship to the literature

Five papers have, in some form or another, characterized the Ramsey steady state in a heterogeneous-agent economy with uninsurable idiosyncratic risk. Before our conclusion, we explain how our results relate to these papers.

Aiyagari (1995) considers an economy with capital (as in section 7) and GHH preferences (as in section 6.2). Aiyagari (1995) allows for endogenous government spending, with some utility function $U(G_t)$ over government spending. The paper assumes that, at the Ramsey steady state, $U'(G_t)$ remains finite. This effectively amounts to assuming a converging Lagrange multiplier λ_t. Aiyagari (1995) uses this to derive the modified golden rule (41). Our results in section 6.2 suggest that the Lagrange multiplier λ_t may not have to converge in a GHH economy. In an economy with endogenous government spending, this corresponds to diverging $U'(G_t)$.[14] As we show in proposition 9, in this case, the modified golden rule fails to hold. Thus, our results qualify those in Aiyagari (1995).

In another theoretical contribution, Chien and Wen (2022) consider additively separable preferences as in (31). They find that, for $\sigma \geq 1$, no Ramsey steady state exists and consumption converges to zero. While we agree that for $\sigma = 1$ such behavior is possible, our results are not always consistent with theirs. For instance, in sections 5.2–5.3, we show that a well-defined Ramsey steady state can exist in the log-separable economy $\sigma = 1$, once income inequality is sufficiently large or the Frisch elasticity is very low; and in section 6.1, we argue that, when $\sigma > 1$ or $\sigma < 1$, well-defined Ramsey steady states are possible.

Dyrda and Pedroni (2023) is the first paper that solves the entire Ramsey plan in an Aiyagari (1995) economy with generalized balanced-growth preferences (10). They achieve this by explicitly parameterizing the paths of the planner's policy variables, and then solving for the optimal combination of all parameters. Our paper has nothing to say about the transition. Our paper is, instead, squarely focused on precisely solving for the long-run Ramsey steady state. For $\sigma > 1$, as assumed by Dyrda and Pedroni (2023), we do not necessarily find immiseration (see section 6.3), though we consistently find near-immiseration tax rates, above 90%. Other differences in the setups between this paper and Dyrda and Pedroni (2023) could explain different findings for long-run taxation.

The first paper to numerically compute the Ramsey steady state in an Aiyagari (1995) economy is Acikgoz et al. (2018). That paper considers a GHH economy with relatively low government spending. Following a primal approach, it derives a set of first order conditions that can be used to study a Ramsey steady state with converging Lagrange multipliers. In this paper, we have developed a complementary approach based on the dual, using it to comprehensively analyze the Ramsey steady state across a wide range of different economies. We have been able to numerically confirm the results in Acikgoz et al. (2018) using our approach with converging multipliers for a

[14]This can happen even if G_t remains positive in the limit, e.g. if $U(G) = \log(G - \underline{G})$ with some subsistence government consumption $\underline{G} > 0$. In a different model but with endogenous spending, Song, Storesletten and Zilibotti (2012, prop. 2) also finds that $G_t \to 0$ is possible in the Ramsey steady state of their model.

GHH economy, as in section 6.2.

Finally, in a recent contribution, LeGrand and Ragot (2023) solve for the RSS in the Woodford (1990) model analytically, allowing for various utility functions, as well as numerically for the RSS in an Aiyagari (1995) model with GHH preferences.[15] As described in appendix 8.3, our results on the Woodford (1990) model are fully consistent with those in LeGrand and Ragot (2023).

10 Conclusion

Our results suggest that immiseration is a widespread long-run outcome of Ramsey plans in standard Aiyagari (1995) economies. In conjunction with the findings in Straub and Werning (2020), this suggests that dynamic Ramsey taxation can lead to extreme long-run behavior in some of the most common workhorse models of household behavior.

This result raises several important questions for future work. First, it would be interesting to investigate if reasonable degrees of limited commitment can give rise to reasonable Ramsey steady states. Second, it would be interesting to extend our analysis to allow for different models of labor supply, e.g. to models with human capital accumulation or indivisibilities in labor supply. Deviations from perfect foresight are also a promising avenue going forward as they may limit the anticipatory labor supply response that are at the core of our immiseration results.

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Optimal Long-Run Fiscal Policy with Heterogeneous Agents — Appendix (Incomplete) —

Adrien Auclert, Michael Cai, Matthew Rognlie and Ludwig Straub

A Proof of proposition 1

In this appendix, we prove proposition 1. Our proof has five steps.

A.1 Smoothness of the value function and the wealth distribution

A.2 Continuity of $\mathcal{A}, \mathcal{C}, \mathcal{N}, \mathcal{U}$

A.3 Differentiability of $\mathcal{A}, \mathcal{C}, \mathcal{N}, \mathcal{U}$

A.4 β-quasi-Toeplitz around steady state

A.5 β-quasi-Toeplitz around converging path

B Other proofs

B.1 Existence of discounted elasticities

In this section, we show that derivatives of the sort introduced in (19) are well-defined. Specifically, let $\mathcal{X}$ be a sequence-space function of sequences $\{r_s, w_s\}$ as in assumption **??**. We show that the discounted derivative

$$\lim_{s\to\infty} \sum_{h=-s}^{\infty} \delta^h \frac{\partial \mathcal{X}_{s+h}}{\partial r_s} \tag{A.1}$$

evaluated around constant paths $r_s = r, w_s = w$, is well-defined for $\delta \in [\beta, 1]$. The result for derivatives w.r.t. w_s follows analogously.

To prove that (A.1) is well-defined, note that because a constant path is obviously a convergent sequence, $\mathcal{X}$ is Fréchet-differentiable by assumption **??**. Represent its derivative by the β-quasi-Toeplitz matrix

$$\mathbf{M} = [M_{t,s}]$$

which we decompose into its Toeplitz portion $\mathbf{T}(\mathbf{a}) \equiv [a_{t-s}]_{t,s=0}^{\infty}$, where $\mathbf{a}$ is the symbol vector as in assumption **??**, and the correction term $\mathbf{E} = [E_{t,s}]$ defined as

$$E_{t,s} = M_{t,s} - a_{t-s}$$

We can thus rewrite the derivative (A.1) as

$$\lim_{s\to\infty}\sum_{h=-s}^{\infty}\delta^h M_{s+h,s} = \lim_{s\to\infty}\sum_{h=-s}^{\infty}\delta^h\left(a_h + E_{s+h,s}\right) \tag{A.2}$$

We next prove that

$$\lim_{s\to\infty}\sum_{h=-s}^{\infty}\delta^h E_{s+h,s} = 0$$

This is clearly true if the stronger statement holds,

$$\lim_{s\to\infty}\sum_{h=-\infty}^{\infty}\delta^h |E_{s+h,s}| = 0 \tag{A.3}$$

where we use the notation $E_{j,s} = 0$ for $j < 0$. The sum in (A.3) is well-defined by (9), for any $\delta \in [\beta, 1]$. The limit of $\sum_{h=-\infty}^{\infty}\delta^h|E_{s+h,s}|$ is zero. This follows from the dominated convergence theorem because (a) $|E_{s+h,s}| \to 0$ converges "pointwise" for each h by equation (8), and (b) because $\delta^h|E_{s+h,s}|$ is dominated by a summable function, independent of s, using (9).

We have established that

$$\lim_{s\to\infty}\sum_{h=-s}^{\infty}\delta^h M_{s+h,s} = \lim_{s\to\infty}\sum_{h=-s}^{\infty}\delta^h a_h \tag{A.4}$$

which exists by equation (**??**) in assumption 1. This proves existence of the discounted elasticities introduced in section (2.4).

B.2 Proof of proposition 2 (Implementability)

If $\{r_t, w_t\}_{t=0}^{\infty}$ are part of a competitive equilibrium, then clearly the goods market clearing condition (15) has to hold, once optimal consumption and labor supply have been substituted in. Likewise, the government budget constraint (16), with optimal asset demand $\mathcal{A}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right)$ and labor supply $\mathcal{N}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right)$, has to hold as well.

We therefore turn our attention to the other direction. Assume sequences $\{r_t, w_t\}_{t=0}^{\infty}$ for which

$$\mathcal{C}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right) + G = \mathcal{N}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right) \tag{A.5}$$

We construct a competitive equilibrium with sequences $\{r_t, w_t\}_{t=0}^{\infty}$. We define: $\tau_t \equiv 1 - w_t$, $C_t \equiv \mathcal{C}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right)$, $B_t \equiv A_t \equiv \mathcal{A}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right)$, $N_t \equiv Y_t \equiv \mathcal{N}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right)$. These objects obviously satisfy conditions 1, 2, 3, and 5 of definition 2.

The only condition remaining is condition 4, the government budget constraint. To derive it, we note that the optimal household policies must satisfy the consolidated household budget constraint

$$\mathcal{C}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right) + \mathcal{A}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right) = (1 + r_t)\,\mathcal{A}_{t-1}\left(\{r_s, w_s\}_{s=0}^{\infty}\right) + w_t\mathcal{N}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right)$$

which is simply (2), integrated across households i. Substituting out $\mathcal{C}_t$ using (A.5), we arrive at

$$G + \mathcal{A}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right) = (1+r_t)\,\mathcal{A}_{t-1}\left(\{r_s, w_s\}_{s=0}^{\infty}\right) + (1-w_t)\,\mathcal{N}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right) \tag{A.6}$$

which is exactly the government budget constraint, condition 4. This shows that (15) is sufficient for a competitive equilibrium. Tracing these same steps backwards, from (A.6) to (A.5) shows that (16) is also sufficient for a competitive equilibrium. This concludes our proof of proposition 2.

B.3 Proof of lemma 1

To prove lemma 1, we need to show both directions. By definition 3, for every Ramsey steady state with prices r, w there is a solution $\{r_s, w_s\}$ of the Ramsey problem such that $w_t \to w$, and $r_t \to r$. This gives us that $\{r_s, w_s\}_{s=0}^{\infty}$ with $r_t \to r$ and $w_t \to w$ is *necessary* for r, w to be part of a Ramsey steady state.

To show that it is also sufficient, take a steady state equilibrium with prices r, w. Suppose there is a Ramsey plan $\{r_s, w_s\}_{s=0}^{\infty}$ with $r_t \to r$ and $w_t \to w$. All we need to show is that $\mathcal{C}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right) \to \mathcal{C}^{ss}(r, w)$, $\mathcal{A}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right) \to \mathcal{A}^{ss}(r, w)$, and $\mathcal{N}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right) \to \mathcal{N}^{ss}(r, w)$. This follows directly from property (ii) of assumption **??**.

B.4 Discounted elasticities of utility when $\delta = \beta$

In this section, we prove that

$$\epsilon^{U,w}(\beta) = \lim_{s\to\infty} \sum_{h=-s}^{\infty} \beta^h \frac{\partial \mathcal{U}_{s+h}}{\partial \log w_s} = w \cdot \int u_c(c_{it}, n_{it}) e_{it} n_{it} di \tag{A.7}$$

and

$$\epsilon^{U,r}(\beta) = \lim_{s\to\infty} \sum_{h=-s}^{\infty} \beta^h \frac{\partial \mathcal{U}_{s+h}}{\partial r_s} = \int u_c(c_{it}, n_{it}) a_{it-1} di \tag{A.8}$$

(22) follows directly from these two equations. We focus on (A.7); (A.8) follows analogously.

To derive (A.7), observe that,

$$\sum_{h=-s}^{\infty} \beta^{s+h} \mathcal{U}_{s+h} = \int \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_{it}, n_{it}) di$$

so that, around a steady state with interest rate r and wage w,

$$\sum_{h=-s}^{\infty} \beta^{s+h} \frac{\partial \mathcal{U}_{s+h}}{\partial \log w_s} = w \cdot \int \frac{\partial \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_{it}, n_{it})}{\partial w_s} di \tag{A.9}$$

Applying the envelope theorem to the household utility maximization problem (1) with budget constraint (2), we find that

$$\frac{\partial \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_{it}, n_{it})}{\partial w_s} = \beta^s \mathbb{E}_0 \left[u_c(c_{is}, n_{is}) \cdot e_{is} n_{is} \right]$$

Substituting this into (A.9), we find

$$\sum_{h=-s}^{\infty} \beta^{s+h} \frac{\partial \mathcal{U}_{s+h}}{\partial \log w_s} = \beta^s w \cdot \int \mathbb{E}_0 \left[u_c(c_{is}, n_{is}) \cdot e_{is} n_{is} \right] di$$

Changing the order of integration, and noting that the cross-sectional average of $u_c(c_{is}, n_{is}) \cdot e_{is} n_{is}$ is deterministic, we have

$$\sum_{h=-s}^{\infty} \beta^{s+h} \frac{\partial \mathcal{U}_{s+h}}{\partial \log w_s} = \beta^s w \cdot \int u_c(c_{is}, n_{is}) \cdot e_{is} n_{is} di$$

This directly implies (A.7), proving the desired result.

B.5 Proof of proposition 3

The Ramsey problem, maximizing (20) subject to implementability (16), admits two necessary first-order conditions (FOCs) that need to be satisfied for any Ramsey plan $\{r_t, w_t\}_{t=0}^{\infty}$: a first-order condition w.r.t. r_s

$$\sum_{h=-s}^{\infty} \delta^h \frac{\partial \mathcal{U}_{s+h}}{\partial r_s} + \sum_{h=-s}^{\infty} \delta^h \lambda_{s+h} \left(\frac{\partial \mathcal{A}_{s+h}}{\partial r_s} + (1 - w_{s+h}) \frac{\partial \mathcal{N}_{s+h}}{\partial r_s} - (1 + r_{s+h}) \frac{\partial \mathcal{A}_{s+h-1}}{\partial r_s} \right) - \lambda_s \mathcal{A}_{s-1} = 0 \tag{A.10}$$

and one w.r.t. w_s,

$$\sum_{h=-s}^{\infty} \delta^h \frac{\partial \mathcal{U}_{s+h}}{\partial w_s} + \sum_{h=-s}^{\infty} \delta^h \lambda_{s+h} \left(\frac{\partial \mathcal{A}_{s+h}}{\partial w_s} + (1 - w_{s+h}) \frac{\partial \mathcal{N}_{s+h}}{\partial w_s} - (1 + r_{s+h}) \frac{\partial \mathcal{A}_{s+h-1}}{\partial w_s} \right) - \lambda_s \mathcal{N}_s = 0 \tag{A.11}$$

Here, λ_t is the current-value multiplier on the implementability constraint (current value w.r.t. discount factor δ). All curly functions are evaluated at the Ramsey plan $\{r_t, w_t\}_{t=0}^{\infty}$. In addition to the two FOCs, the implementability condition (16) itself, of course, also has to hold along the Ramsey plan.

Now consider the situation assumed in proposition 3. The Ramsey plan $\{r_t, w_t\}_{t=0}^{\infty}$ is assumed to converge to a pair of prices (r, w); and the multiplier λ_t is assumed to converge, too, to some value λ. If $\lambda = 0$, the RSS must be at the unconstrained optimum, that is,

$$\sum_{h=-s}^{\infty} \delta^h \frac{\partial \mathcal{U}_{s+h}}{\partial r_s} \to 0 \qquad \sum_{h=-s}^{\infty} \delta^h \frac{\partial \mathcal{U}_{s+h}}{\partial w_s} \to 0$$

To derive (25), we instead work with the case where $\lambda \neq 0$.

By lemma 1, we know that convergence of $\{r_t, w_t\}_{t=0}^{\infty}$ implies that all the curly functions converge to their steady state values, evaluated at (r, w). In particular, the implementability condition (16),

$$G + (1 + r_t)\,\mathcal{A}_{t-1}\left(\{r_s, w_s\}_{s=0}^{\infty}\right) = \mathcal{A}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right) + (1 - w_t)\,\mathcal{N}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right)$$

must converge, as $t \to \infty$, to the steady state government budget constraint

$$G + r\mathcal{A}^{ss}(r, w) = (1 - w)\,\mathcal{N}^{ss}(r, w)$$

which is exactly (24).

We begin deriving (25) by taking limits of the FOCs (A.10) and (A.11) as $s \to \infty$ and by simplifying the resulting expressions. Since these steps are almost exactly analogous for both FOCs, we focus on (A.10).

To take the limit of (A.10) as $s \to \infty$, we first need to ensure that the limits exist. The first term clearly has a limit,

$$\sum_{h=-s}^{\infty} \delta^h \frac{\partial \mathcal{U}_{s+h}}{\partial r_s} \to \epsilon^{U,r}(\delta) \qquad \text{as } s \to \infty$$

The last term does, too, $\lambda_s \mathcal{A}_{s-1} \to \lambda \mathcal{A}^{ss}(r, w)$. Next consider the expression

$$\sum_{h=-s}^{\infty} \delta^h \lambda_{s+h} \frac{\partial \mathcal{A}_{s+h}}{\partial r_s}$$

which appears in the middle term in (A.10). Recall that matrix $\left[\frac{\partial \mathcal{A}_t}{\partial r_s}\right]_{t,s}$ is β-quasi-Toeplitz. Denote by $\mathbf{a} = (a_t)$ the symbol vector of quasi-Toeplitz matrix $\left[\frac{\partial \mathcal{A}_t}{\partial r_s}\right]_{t,s}$ and by $E_{t,s} \equiv \frac{\partial \mathcal{A}_t}{\partial r_s} - a_{t,s}$ its correction matrix. Define the matrix $M_{t,s} \equiv \lambda_t \frac{\partial \mathcal{A}_t}{\partial r_s}$. $M_{t,s}$ is quasi-Toeplitz as $\lim_{u\to\infty} M_{t+u,s+u} = \lambda a_{t-s}$. Moreover, the tails of λa_t are clearly bounded exactly as in (**??**), and the tails of the correction $M_{t,s} - \lambda a_{t-s} = (1 - \lambda)\,a_{t-s} + E_{t,s}$ are bounded as in (9). This establishes that $M_{t,s}$ is β-quasi-Toeplitz, as in definition 1. Thus, following the reasoning in section B.1 (specifically (A.4), which holds for arbitrary β-quasi-Toeplitz matrices), we have that

$$\sum_{h=-s}^{\infty} \delta^h \lambda_{s+h} \frac{\partial \mathcal{A}_{s+h}}{\partial r_s} \to \sum_{h=-\infty}^{\infty} \delta^h \lambda a_h = \lambda \sum_{h=-\infty}^{\infty} \delta^h a_h = \lambda \mathcal{A}^{ss}(r, w) \epsilon^{A,r}(\delta)$$

as $s \to \infty$. Analogous steps show that

$$\sum_{h=-s}^{\infty} \delta^h \lambda_{s+h} (1 - w_{s+h}) \frac{\partial \mathcal{N}_{s+h}}{\partial r_s} \to \lambda (1 - w)\,\mathcal{N}^{ss}(r, w) \epsilon^{N,r}(\delta)$$

and

$$\sum_{h=-s}^{\infty} \delta^h \lambda_{s+h} (1 + r_{s+h}) \frac{\partial \mathcal{A}_{s+h-1}}{\partial r_s} \to \lambda \delta (1 + r)\,\mathcal{A}^{ss}(r, w) \epsilon^{A,r}(\delta)$$

as $s \to \infty$.

Taken together, the preceding limit analysis establishes that a limit of all the terms in the FOC w.r.t. r_s (A.10) exists, with the limiting equation

$$\epsilon^{U,r}(\delta) + \lambda \left(A\epsilon^{A,r}(\delta) + (1-w)\, N\epsilon^{N,r}(\delta) - \delta\,(1+r)\, A\epsilon^{A,r}\,(\delta) \right) - \lambda A = 0 \tag{A.12}$$

where we abbreviate assets and labor supply at the Ramsey steady state as $A = \mathcal{A}^{ss}(r,w)$ and $N = \mathcal{N}^{ss}(r,w)$. The limit of the FOC w.r.t. w_s (A.11) can similarly found to be[A-1]

$$\epsilon^{U,w}(\delta) + \lambda \left(A\epsilon^{A,w}(\delta) + (1-w)\, N\epsilon^{N,w}(\delta) - \delta\,(1+r)\, A\epsilon^{A,w}\,(\delta) \right) - \lambda N w = 0 \tag{A.13}$$

To derive (25), we use our definition for liquidity, $\ell \equiv \frac{A}{wN}$, and simplify (A.12) and (A.13), obtaining

$$\lambda^{-1}A^{-1}\ell\epsilon^{U,r}(\delta) + (1-\delta\,(1+r))\,\ell\epsilon^{A,r}(\delta) + \frac{1-w}{w}\epsilon^{N,r}(\delta) - \ell = 0 \tag{A.14}$$

$$\lambda^{-1}A^{-1}\ell\epsilon^{U,w}(\delta) + (1-\delta\,(1+r))\,\ell\epsilon^{A,w}(\delta) + \frac{1-w}{w}\epsilon^{N,w}(\delta) - 1 = 0 \tag{A.15}$$

After a few steps of algebra, our second Ramsey steady state condition (25) can be derived by eliminating λ from those two equations. This proves proposition 3.

B.6 Proof of proposition 4

To prove the proposition, we build on the following lemma, which itself extends proposition 8 in Auclert et al. (2024a).

Lemma 3. *Assume $u(c,n)$ is log-separable, as in (11). Fix a date $\iota > 0$ and some $\chi > 0$. Fix sequences $\{r_s, w_s\}$ for which $\mathcal{A}, \mathcal{N}, \mathcal{C}, \mathcal{U}$ are well-defined. Construct new sequences $\{\tilde{r}_s, \tilde{w}_s\}$*

$$1 + \tilde{r}_s \equiv \begin{cases} (1+r_s)\,\chi & \text{if } s = \iota \\ (1+r_s)\,/\chi & \text{if } s = \iota + 1 \\ 1 + r_s & \text{if } s \notin \{\iota, \iota+1\} \end{cases}$$

and

$$\tilde{w}_s = \begin{cases} w_s \chi & \text{if } s = \iota \\ w_s & \text{if } s \neq \iota \end{cases}$$

Then, $\mathcal{A}, \mathcal{N}, \mathcal{C}, \mathcal{U}$ evaluated at the new sequences are given by

$$\mathcal{A}_t\left(\{\tilde{r}_s, \tilde{w}_s\}\right) = \begin{cases} \chi \mathcal{A}_t\left(\{r_s, w_s\}\right) & \text{if } t = \iota \\ \mathcal{A}_t\left(\{r_s, w_s\}\right) & \text{if } t \neq \iota \end{cases} \tag{A.16}$$

[A-1] The w in the final term stems from the fact that elasticities w.r.t. w are effectively derivatives w.r.t. $\log w$ instead of w.

$$\mathcal{C}_t\left(\{\tilde{r}_s,\tilde{w}_s\}\right)=\begin{cases}\chi\mathcal{C}_t\left(\{r_s,w_s\}\right) & \text{if } t=\iota\\ \mathcal{C}_t\left(\{r_s,w_s\}\right) & \text{if } t\neq\iota\end{cases} \tag{A.17}$$

$$\mathcal{U}_t\left(\{\tilde{r}_s,\tilde{w}_s\}\right)=\begin{cases}\log\chi+\mathcal{U}_t\left(\{r_s,w_s\}\right) & \text{if } t=\iota\\ \mathcal{U}_t\left(\{r_s,w_s\}\right) & \text{if } t\neq\iota\end{cases} \tag{A.18}$$

$$\mathcal{N}_t\left(\{\tilde{r}_s,\tilde{w}_s\}\right)=\mathcal{N}_t\left(\{r_s,w_s\}\right) \tag{A.19}$$

Proof. Consider the household problem with the old sequences $\{r_s, w_s\}$,

$$\max\ \mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\log c_{it}-v(n_{it})\right] \tag{A.20}$$

subject to

$$c_{it}+a_{it}=(1+r_t)\,a_{it-1}+w_te_{it}n_{it}$$

and $a_{it}\geq 0$. Let c_t^*, a_t^*, n_t^* denote the optimal policies for those sequences, and Ψ_t^* denote the associated income and wealth distributions. Next, consider the problem with the new sequences $\{\tilde{r}_s, \tilde{w}_s\}$,

$$\max\ \mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\log c_{it}-v(n_{it})\right] \tag{A.21}$$

subject to

$$c_{it}+a_{it}=(1+\tilde{r}_t)\,a_{it-1}+\tilde{w}_te_{it}n_{it}$$

and $a_{it}\geq 0$. We rewrite this problem by transforming the variables c_{it}, a_{it}, n_{it} into $\hat{c}_{it}, \hat{a}_{it}, \hat{n}_{it}$ as follows

$$\hat{c}_{it}=\begin{cases}c_{it}/\chi & \text{if } t=\iota\\ c_{it} & \text{if } t\neq\iota\end{cases}$$

$$\hat{a}_{it}=\begin{cases}a_{it}/\chi & \text{if } t=\iota\\ a_{it} & \text{if } t\neq\iota\end{cases}$$

$$\hat{n}_{it}=n_{it}$$

It is straightforward to verify that, by construction, $\hat{c}_{it}, \hat{a}_{it}, \hat{n}_{it}$ satisfy the budget constraint with the original sequences,

$$\hat{c}_{it}+\hat{a}_{it}=(1+r_t)\,\hat{a}_{it-1}+w_te_{it}\hat{n}_{it}$$

Moreover, the objective evaluated at $\hat{c}_{it}, \hat{a}_{it}, \hat{n}_{it}$ just shifts by a constant,

$$\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\log\hat{c}_{it}-v(\hat{n}_{it})\right]=\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\log c_{it}-v(n_{it})\right]-\beta^\iota\log\chi$$

Thus, for any solution c_{it}, a_{it}, n_{it} to the utility maximization problem (A.21) with the new sequences the tuple $\hat{c}_{it}, \hat{a}_{it}, \hat{n}_{it}$ satisfies the utility maximization problem (A.20) at the original sequences. This implies that the policy functions associated with (A.21) are simply given by

$$\tilde{c}_t^*(e, a_-) = \begin{cases} \chi c_t^*(e, a_-) & \text{if } t = \iota \\ c_t^*(e, \chi^{-1} a_-) & \text{if } t = \iota + 1 \\ c_t^*(e, a_-) & \text{if } t \notin \{\iota, \iota + 1\} \end{cases}$$

$$\tilde{a}_t^*(e, a_-) = \begin{cases} \chi a_t^*(e, a_-) & \text{if } t = \iota \\ a_t^*(e, \chi^{-1} a_-) & \text{if } t = \iota + 1 \\ a_t^*(e, a_-) & \text{if } t \notin \{\iota, \iota + 1\} \end{cases}$$

$$\tilde{n}_t^*(e, a_-) = \begin{cases} n_t^*(e, a_-) & \text{if } t = \iota \\ n_t^*(e, \chi^{-1} a_-) & \text{if } t = \iota + 1 \\ n_t^*(e, a_-) & \text{if } t \notin \{\iota, \iota + 1\} \end{cases}$$

and the cumulative wealth distribution in income state e is given by

$$\tilde{\Psi}_t^*(e, a_-) = \begin{cases} \Psi_t^*(e, \chi^{-1} a_-) & \text{if } t = \iota + 1 \\ \Psi_t^*(e, a_-) & \text{if } t \neq \iota \end{cases}$$

Substituting these equations into (4), (5), (6), and (7) then immediately gives us (A.16)—(A.19). □

Lemma 3 is very useful because it relates household behavior in response interest rate changes with household behavior in response to wage changes. It relies on the combination of balanced growth and a unitary elasticity of intertemporal substitution. The following lemma follows straight from lemma 3.

Lemma 4. *Assume $u(c, n)$ is log-separable, as in (11). Fix a steady state r, w. Then,*

$$\epsilon^{A,w}(\delta) + (1 - \delta)(1 + r)\,\epsilon^{A,r}(\delta) = 1 \tag{A.22}$$

$$\epsilon^{C,w}(\delta) + (1 - \delta)(1 + r)\,\epsilon^{C,r}(\delta) = 1$$

$$\epsilon^{U,w}(\delta) + (1 - \delta)(1 + r)\,\epsilon^{U,r}(\delta) = 1$$

$$\epsilon^{N,w}(\delta) + (1 - \delta)(1 + r)\,\epsilon^{N,r}(\delta) = 0$$

Proof. We prove (A.22). The other equations are derived very similarly. Differentiating (A.16) with respect to χ around a steady state with constant r, w and around $\chi = 1$, we find

$$\frac{\partial \mathcal{A}_t}{\partial w_\iota} w + \left(\frac{\partial \mathcal{A}_t}{\partial r_\iota} - \frac{\partial \mathcal{A}_t}{\partial r_{\iota+1}} \right)(1 + r) = A 1_{\{\iota = t\}}$$

where $A = \mathcal{A}^{ss}(r, w)$. Discounting, we obtain

$$\delta^h \frac{\partial \log \mathcal{A}_{\iota+h}}{\partial \log w_\iota} + \left(\delta^h \frac{\partial \log \mathcal{A}_{\iota+h}}{\partial r_\iota} - \delta^h \frac{\partial \log \mathcal{A}_{\iota+h}}{\partial r_{\iota+1}} \right) (1+r) = \delta^h 1_{\{h=0\}}$$

Summing and taking limits $\iota \to \infty$, this becomes

$$\epsilon^{A,w}(\delta) + (1-\delta)(1+r)\,\epsilon^{A,r} = 1$$

confirming (A.22). □

To prove equation (26) in proposition 4, we now begin from the first order conditions (A.14) and (A.15) in the proof of proposition 3,

$$\lambda^{-1} A^{-1} \ell \epsilon^{U,r}(\delta) + (1 - \delta(1+r))\, \ell \epsilon^{A,r}(\delta) + \frac{1-w}{w} \epsilon^{N,r}(\delta) - \ell = 0$$

$$\lambda^{-1} A^{-1} \ell \epsilon^{U,w}(\delta) + (1 - \delta(1+r))\, \ell \epsilon^{A,w}(\delta) + \frac{1-w}{w} \epsilon^{N,w}(\delta) - 1 = 0$$

Now sum 1 times the first equation with $(1-\delta)(1+r)$ times the second equation. After some algebra, this precisely gives

$$\lambda = \frac{1}{rA + wN} = \frac{1}{C} \tag{A.23}$$

which is very intuitive as $1/C$ is the marginal utility of an agent consuming aggregate consumption. Substituting this result back into (A.14), we have

$$(1 - \delta(1+r))\, \ell \epsilon^{A,r}(\delta) + \frac{1-w}{w} \epsilon^{N,r}(\delta) + (r\ell + 1)\, \epsilon^{U,r}(\delta) - \ell = 0$$

which is precisely (26).

B.7 Proof of proposition 5

As in the proof of proposition 3, we begin with FOCs (A.10) and (A.11). The main difference between propositions 3 and 5 is that in proposition 5, we assume the Lagrange multiplier to diverge, $\lambda_t \to \pm\infty$, with a well-defined limiting growth rate

$$\eta \equiv \lim_{t \to \infty} \frac{\lambda_t}{\lambda_{t-1}} \in [1, \delta^{-1}]$$

Just like before, we focus our attention on the FOC w.r.t. r_s (A.10).

To still be able to take limits of the FOC, we now first divide the FOC by λ_s,

$$\lambda_s^{-1} \sum_{h=-s}^{\infty} \delta^h \frac{\partial \mathcal{U}_{s+h}}{\partial r_s} + \sum_{h=-s}^{\infty} \delta^h \frac{\lambda_{s+h}}{\lambda_s} \left(\frac{\partial \mathcal{A}_{s+h}}{\partial r_s} + (1 - w_{s+h}) \frac{\partial \mathcal{N}_{s+h}}{\partial r_s} - (1 + r_{s+h}) \frac{\partial \mathcal{A}_{s+h-1}}{\partial r_s} \right) - \mathcal{A}_{s-1} = 0 \tag{A.24}$$

This is the expression for which we take limits $s \to \infty$. The first term of this modified FOC converges to zero. The last term converges to $\mathcal{A}^{ss}(r, w)$. The crux is again disciplining the terms in the middle. We show how to do this for the first of the terms in the middle,

$$\sum_{h=-s}^{\infty} \delta^h \frac{\lambda_{s+h}}{\lambda_s} \frac{\partial \mathcal{A}_{s+h}}{\partial r_s}$$

Like before, recall that the matrix $\left[\frac{\partial \mathcal{A}_t}{\partial r_s} \right]$ is β-quasi-Toeplitz, with some symbol vector $\mathbf{a} = (a_t)$ and correction matrix $E_{t,s} = \frac{\partial \mathcal{A}_t}{\partial r_s} - a_{t-s}$.

We now define the matrix $\mathbf{M}$ as

$$M_{t,s} \equiv \frac{\lambda_t}{\eta^{t-s} \lambda_s} \frac{\partial \mathcal{A}_t}{\partial r_s}$$

The matrix is quasi-Toeplitz with symbol vector $\mathbf{a}$, because

$$M_{t+u,s+u} = \underbrace{\frac{\lambda_{t+u}}{\eta^{t-s} \lambda_{s+u}}}_{\to 1} \frac{\partial \mathcal{A}_{t+u}}{\partial r_{s+u}} \to a_{t-s}$$

Moreover, $\mathbf{a}$ has exponential tails as in (**??**). Finally, consider

$$M_{t,s} - a_{t-s} = \frac{\lambda_t}{\eta^{t-s} \lambda_s} \frac{\partial \mathcal{A}_t}{\partial r_s} - a_{t-s} = \frac{\lambda_t}{\eta^{t-s} \lambda_s} \left(\frac{\partial \mathcal{A}_t}{\partial r_s} - a_{t-s} \right) - \left(1 - \frac{\lambda_t}{\eta^{t-s} \lambda_s} \right) a_{t-s}$$

and so

$$|M_{t,s} - a_{t-s}| \leq \left| \frac{\lambda_t}{\eta^{t-s} \lambda_s} \right| |E_{t-s}| + \left(1 + \left| \frac{\lambda_t}{\eta^{t-s} \lambda_s} \right| \right) |a_{t-s}|$$

We can use this expression to bound $|M_{t,s} - a_{t-s}|$ above for $t - s \geq 0$ as

$$|M_{t,s} - a_{t-s}| \leq \left| \frac{\lambda_t}{\eta^{t-s} \lambda_s} \right| C_3 \tilde{\gamma}^{t-s} + \left(1 + \left| \frac{\lambda_t}{\eta^{t-s} \lambda_s} \right| \right) C_1 \tilde{\gamma}^{t-s} \tag{A.25}$$

where we use the same notation as in (**??**) and (9). Pick a $\hat{\gamma} \in (\tilde{\gamma}, 1)$. This means that $\tilde{\gamma}/\hat{\gamma} < 1$. Now observe that

$$\frac{\lambda_t}{\eta^{t-s} \lambda_s} \times \left(\frac{\tilde{\gamma}}{\hat{\gamma}} \right)^{t-s}$$

must be bounded above by some $C_5 > 0$, uniformly for all t, s with $t \geq s$. If it wasn't, then there have to be subsequences $s_1, s_2, \ldots$ and $t_1, t_2, \ldots$ such that $t_i \geq s_i$ for all i and

$$\frac{\lambda_{t_i}}{\eta^{t_i - s_i} \lambda_{s_i}} \times \left(\frac{\tilde{\gamma}}{\hat{\gamma}} \right)^{t_i - s_i} \to \infty \tag{A.26}$$

This, however, is only possible for there is a separate subsequence (t_i'), such that between $t_i' - 1$ and t_i, the growth rate of λ_t lies above $\eta \frac{\hat{\gamma}}{\tilde{\gamma}}$,

$$\frac{\lambda_{t_i}}{\lambda_{t_i - 1}} \geq \eta \frac{\hat{\gamma}}{\tilde{\gamma}}$$

If there was no such subsequence (t_i'), then $\frac{\lambda_{t_i}}{\eta^{t_i - s_i} \lambda_{s_i}} \times \left(\frac{\tilde{\gamma}}{\hat{\gamma}} \right)^{t_i - s_i}$ would always be bounded above by 1, contradicting (A.26). But $\frac{\lambda_{t_i}}{\lambda_{t_i - 1}} \geq \eta \frac{\hat{\gamma}}{\tilde{\gamma}}$ contradicts the assumption of $\lim_{t \to \infty} \frac{\lambda_t}{\lambda_{t-1}} = \eta$. Therefore, $\frac{\lambda_t}{\eta^{t-s} \lambda_s} \times \left(\frac{\tilde{\gamma}}{\hat{\gamma}} \right)^{t-s}$ must indeed be bounded by some $C_5 > 0$. Having established this, we rewrite the bound in (A.25) as

$$|M_{t,s} - a_{t-s}| \leq \underbrace{\left| \frac{\lambda_t}{\eta^{t-s} \lambda_s} \right| \left(\frac{\tilde{\gamma}}{\hat{\gamma}} \right)^{t-s}}_{\leq C_5} C_3 \hat{\gamma}^{t-s} + \underbrace{\left(\left(\frac{\tilde{\gamma}}{\hat{\gamma}} \right)^{t-s} + \left| \frac{\lambda_t}{\eta^{t-s} \lambda_s} \right| \left(\frac{\tilde{\gamma}}{\hat{\gamma}} \right)^{t-s} \right)}_{\leq 1 + C_5} C_1 \hat{\gamma}^{t-s}$$

Following analogous steps for indices $t < s$, this shows that the matrix $\mathbf{M} = [M_{t,s}]$ is β-quasi-Toeplitz, as in definition 1.

With matrix $\mathbf{M}$ being β-quasi-Toeplitz with symbol vector $\mathbf{a}$, we have that

$$\sum_{h=-s}^{\infty} \delta^h \frac{\lambda_{s+h}}{\lambda_s} \frac{\partial \mathcal{A}_{s+h}}{\partial r_s} = \sum_{h=-s}^{\infty} (\delta \eta)^h M_{s+h,s} \to \sum_{h=-\infty}^{\infty} (\delta \eta)^h a_h = \mathcal{A}^{ss}(r, w) \epsilon^{A,r}(\delta \eta)$$

Following the same steps for the other terms in the middle of (A.24) we can take limits of the FOC as $s \to \infty$, obtaining

$$A \epsilon^{A,r}(\delta \eta) + (1 - w) N \epsilon^{N,r} - \delta \eta (1 + r) A \epsilon^{A,r} - A = 0$$

or simplified,

$$(1 - \delta \eta (1 + r)) \ell \epsilon^{A,r}(\delta \eta) - \frac{1 - w}{w} \left(-\epsilon^{N,r}(\delta \eta) \right) - \ell = 0$$

which is exactly identical to (28). As before, we denote by A and N the steady state values $\mathcal{A}^{ss}(r, w)$ and $\mathcal{N}^{ss}(r, w)$.

Following the same process for the FOC w.r.t. w_s (A.11), we find

$$A \epsilon^{A,w}(\delta \eta) + (1 - w) N \epsilon^{N,w} - \delta \eta (1 + r) A \epsilon^{A,w} - N w = 0$$

Simplifying this expression yields

$$\left(1-\delta\eta\left(1+r\right)\right)\ell\epsilon^{A,w}(\delta\eta)+\frac{1-w}{w}\epsilon^{N,w}-1=0$$

Going from elasticities w.r.t. the wage to elasticities w.r.t. the tax rate, we find

$$\left(1-\delta\eta\left(1+r\right)\right)\ell\epsilon^{A,\tau}(\delta\eta)+\frac{1-w}{w}\epsilon^{N,\tau}+1=0$$

which is identical to (27).

B.8 Proof of proposition 6

Let $\{r_t, w_t\}$ be an optimal Ramsey plan such that $r_t \to r < 1/\beta - 1$ and $w_t \to 0$ with $w_t/w_{t-1} \to \gamma \in [\delta, 1)$. The main idea is to first de-trend the path of after-tax wages w_t, and then take limits of the FOCs as before.

The following lemma describes how de-trending works for the curly sequence-space functions.

Lemma 5. *Assume $u(c,n)$ is balanced growth compatible, as in (10). For any sequences $\{r_t, w_t\}$ and $\gamma \in (0,1)$, we have that*

$$\mathcal{U}_t\left(\{r_s,w_s\},\beta\right)=\gamma^{(1-\sigma)t}\mathcal{U}_t\left(\left\{\left(1+r_s\right)/\gamma-1,w_s\gamma^{-s}\right\},\beta\gamma^{1-\sigma}\right) \tag{A.27}$$

$$\mathcal{A}_t\left(\{r_s,w_s\},\beta\right)=\gamma^{t}\mathcal{A}_t\left(\left\{\left(1+r_s\right)/\gamma-1,w_s\gamma^{-s}\right\},\beta\gamma^{1-\sigma}\right) \tag{A.28}$$

$$\mathcal{N}_t\left(\{r_s,w_s\},\beta\right)=\mathcal{N}_t\left(\left\{\left(1+r_s\right)/\gamma-1,w_s\gamma^{-s}\right\},\beta\gamma^{1-\sigma}\right) \tag{A.29}$$

where the additional argument in the curly functions is the discount factor used in the underlying household preferences (1).

Proof. To prove the lemma, we re-write the household problem that is aggregated by the curly functions $\mathcal{U}, \mathcal{A}, \mathcal{N}$,

$$\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t u(c_{it},n_{it})\right]$$

subject to the budget constraint

$$c_{it}+a_{it}=\left(1+r_t\right)a_{it-1}+w_t e_{it} n_{it}$$

and the borrowing constraint $a_{it} \geq 0$. The utility function $u(c,n)$ is of the KPR form, as in (10)

Now define a "de-trended" version of this problem, where

$$\hat{c}_{it}\equiv c_{it}/\gamma^t, \qquad \hat{a}_{it}\equiv a_{it}/\gamma^t$$

are de-trended consumption and asset holdings. It is straightforward to see that the above utility maximization problem is equivalent to maximizing

$$\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\gamma^{(1-\sigma)t}u(\hat{c}_{it}, n_{it})\right]$$

subject to the budget constraint

$$\hat{c}_{it}+\hat{a}_{it}=\frac{1+r_t}{\gamma}\hat{a}_{it-1}+\gamma^{-t}w_t e_{it} n_{it}$$

and the borrowing constraint $\hat{a}_{it}\geq 0$. This immediately implies the relationships (A.27)–(A.29) stated in the lemma. □

Armed with lemma 5, we now consider the FOCs (A.10) and (A.11). For simplicity, again, we focus first on (A.10). We de-trend (A.10) using lemma 5. To avoid lengthy expressions, we define

$$1+\hat{r}_s\equiv\frac{1+r_s}{\gamma}\qquad 1+\hat{r}\equiv\frac{1+r}{\gamma}\qquad \hat{w}_s\equiv w_s\gamma^{-s}$$

and denote all sequences-space functions with a hat, $\hat{\mathcal{X}}$ if they are evaluated around the de-trended sequences $\{\hat{r}_s,\hat{w}_s\}$. We thus obtain

$$\sum_{h=-s}^{\infty}\delta^h\frac{\partial\hat{\mathcal{U}}_{s+h}}{\partial r_s}+\sum_{h=-s}^{\infty}\delta^h\lambda_{s+h}\left(\gamma^{s+h}\frac{\partial\hat{\mathcal{A}}_{s+h}}{\partial r_s}+\left(1-\hat{w}_{s+h}\gamma^{s+h}\right)\frac{\partial\hat{\mathcal{N}}_{s+h}}{\partial r_s}-\left(1+r_{s+h}\right)\gamma^{s+h-1}\frac{\partial\hat{\mathcal{A}}_{s+h-1}}{\partial r_s}\right)$$
$$-\lambda_s\gamma^{s-1}\hat{\mathcal{A}}_{s-1}=0$$

Dividing by λ_s, this becomes

$$\frac{1}{\lambda_s}\sum_{h=-s}^{\infty}\delta^h\frac{\partial\hat{\mathcal{U}}_{s+h}}{\partial r_s}+\sum_{h=-s}^{\infty}\delta^h\frac{\lambda_{s+h}}{\lambda_s}\left(\gamma^{s+h}\frac{\partial\hat{\mathcal{A}}_{s+h}}{\partial r_s}+\left(1-\hat{w}_{s+h}\gamma^{s+h}\right)\frac{\partial\hat{\mathcal{N}}_{s+h}}{\partial r_s}-\left(1+\hat{r}_{s+h}\right)\gamma^{s+h}\frac{\partial\hat{\mathcal{A}}_{s+h-1}}{\partial r_s}\right)$$
$$-\gamma^{s-1}\hat{\mathcal{A}}_{s-1}=0$$

Observe that, as $s\rightarrow\infty$, all terms converge to zero except for the $\frac{\partial\hat{\mathcal{N}}_{s+h}}{\partial r_s}$ term, which approaches

$$\sum_{h=-\infty}^{\infty}\delta^h\eta^h\frac{\partial\hat{\mathcal{N}}_{s+h}}{\partial r_s}=0$$

or in other words,

$$\epsilon^{N,r}(\delta\eta)=0$$

when evaluated around a de-trended steady state with $\hat{w}=1$ and $1+\hat{r}=(1+r)/\gamma$. Using the same logic, we find that (A.11) implies that

$$\epsilon^{N,\tau}(\delta\eta) = 0$$

Finally, the government budget constraint (16) is given by

$$G + \gamma\left(1+\hat{r}_t\right)\gamma^{t-1}\mathcal{A}_{t-1}\left(\{\hat{r}_s, \hat{w}_s\}_{s=0}^{\infty}\right) = \gamma^t\mathcal{A}_t\left(\{\hat{r}_s, \hat{w}_s\}_{s=0}^{\infty}\right) + \left(1-\gamma^t\hat{w}_t\right)\mathcal{N}_t\left(\{\hat{r}_s, \hat{w}_s\}_{s=0}^{\infty}\right)$$

Taking the limit of $t \to \infty$, we find

$$G = \mathcal{N}^{ss}(\hat{r}, \hat{w})$$

This proves proposition 6.

B.9 Proof of proposition 7

If $\{r_t, w_t, K_t\}_{t=0}^{\infty}$ are part of a competitive equilibrium with capital, then clearly the goods market clearing condition (36) has to hold, once optimal consumption and labor supply have been substituted in. The government budget constraint with capital (35) is given by

$$G + (1+r_t)\,B_{t-1} = B_t + \tau_t w_t^* N_t + \tau_t^k r_t^* K_{t-1}$$

Rearranging and substituting in asset market clearing, we find

$$G + (1+r_t)\left(A_{t-1} - K_{t-1}\right) = A_t - K_t + w_t^* N_t + r_t^* K_{t-1} - w_t N_t - r_t K_{t-1}$$

Using Euler's theorem, we have

$$G + K_t - (1-\delta_k)\,K_{t-1} + (1+r_t)\,A_{t-1} = A_t + K_{t-1}^{\alpha} N_t^{1-\alpha} - w_t N_t$$

Substituting in household behavior for A_t, A_{t-1}, N_t yields the second implementability condition (37).

We therefore turn our attention to the other direction. Assume sequences $\{r_t, w_t, K_t\}_{t=0}^{\infty}$ for which

$$\mathcal{C}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right) + K_t - (1-\delta_k)\,K_{t-1} + G = K_{t-1}^{\alpha}\mathcal{N}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right)^{1-\alpha} \tag{A.30}$$

We construct a competitive equilibrium with sequences $\{r_t, w_t, K_t\}_{t=0}^{\infty}$. We define $N_t = \mathcal{N}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right)$, $Y_t = K_{t-1}^{\alpha} N_t^{1-\alpha}$, $w_t^* = (1-\alpha)\,Y_t/N_t$, $C_t = \mathcal{C}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right)$, $A_t = \mathcal{A}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right)$, $B_t = A_t - K_t$, $\tau_t = 1 - \frac{w_t}{w_t^*}$, $r_t^* = \alpha\left(\frac{N_t}{K_t}\right)^{1-\alpha} - \delta_k$, $\tau_t^k = 1 - \frac{r_t}{r_t^*}$. These objects obviously satisfy conditions 1, 2, 3, and 5 of definition 4.

The only condition remaining is condition 4, the government budget constraint. To derive it, we note that the optimal household policies must satisfy the consolidated household budget constraint

$$\mathcal{C}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right) + \mathcal{A}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right) = (1+r_t)\,\mathcal{A}_{t-1}\left(\{r_s, w_s\}_{s=0}^{\infty}\right) + w_t\mathcal{N}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right)$$

which is simply (2), integrated across households i. Substituting out $\mathcal{C}_t$ using (A.30), we arrive at (37)

$$G + K_t - (1-\delta_k)\,K_{t-1} + (1+r_t)\,\mathcal{A}_{t-1}\left(\{r_s, w_s\}_{s=0}^{\infty}\right) = \mathcal{A}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right) + K_{t-1}^{\alpha}\mathcal{N}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right)^{1-\alpha} - w_t\mathcal{N}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right)$$

which is exactly the government budget constraint, condition 4. This shows that (36) is sufficient for a competitive equilibrium. Tracing these same steps backwards, from (37) to (A.30) shows that (37) is also sufficient for a competitive equilibrium. This concludes our proof of proposition 7.

B.10 Proof of lemma 2

To prove lemma 2, we need to show both directions. By definition 5, for every Ramsey steady state with prices r, w, K there is a solution $\{r_s, w_s, K_s\}$ of the Ramsey problem such that $w_t \to w$, $r_t \to r$, and $K_t \to K$. This gives us that $\{r_s, w_s, K_s\}_{s=0}^{\infty}$ with $r_t \to r$, $w_t \to w$, and $K_t \to K$ is *necessary* for r, w, K to be part of a Ramsey steady state.

To show that it is also sufficient, take a steady state equilibrium with prices r, w, K. Suppose there is a Ramsey plan $\{r_s, w_s, K_s\}_{s=0}^{\infty}$ with $r_t \to r$, $w_t \to w$, and $K_t \to K$. All we need to show is that $\mathcal{C}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right) \to \mathcal{C}^{ss}(r, w)$, $\mathcal{A}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right) \to \mathcal{A}^{ss}(r, w)$, and $\mathcal{N}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right) \to \mathcal{N}^{ss}(r, w)$. This follows directly from property (ii) of assumption **??**.

B.11 Proof of proposition 8

The Ramsey problem, maximizing (20) subject to implementability (37), admits three necessary first-order conditions (FOCs) that need to be satisfied for any Ramsey plan $\{r_t, w_t, K_t\}_{t=0}^{\infty}$: a first-order condition w.r.t. r_s

$$\sum_{h=-s}^{\infty} \delta^h \frac{\partial \mathcal{U}_{s+h}}{\partial r_s} + \sum_{h=-s}^{\infty} \delta^h \lambda_{s+h}\left(\frac{\partial \mathcal{A}_{s+h}}{\partial r_s} + \left(w_{s+h}^* - w_{s+h}\right)\frac{\partial \mathcal{N}_{s+h}}{\partial r_s} - (1+r_{s+h})\frac{\partial \mathcal{A}_{s+h-1}}{\partial r_s}\right) - \lambda_s \mathcal{A}_{s-1} = 0 \tag{A.31}$$

where $w_t^* = (1-\alpha)\,K_{t-1}^{\alpha}\mathcal{N}_t^{-\alpha}$; a first-order condition w.r.t. w_s,

$$\sum_{h=-s}^{\infty} \delta^h \frac{\partial \mathcal{U}_{s+h}}{\partial w_s} + \sum_{h=-s}^{\infty} \delta^h \lambda_{s+h}\left(\frac{\partial \mathcal{A}_{s+h}}{\partial w_s} + \left(w_{s+h}^* - w_{s+h}\right)\frac{\partial \mathcal{N}_{s+h}}{\partial w_s} - (1+r_{s+h})\frac{\partial \mathcal{A}_{s+h-1}}{\partial w_s}\right) - \lambda_s \mathcal{N}_s = 0 \tag{A.32}$$

and a first-order condition w.r.t. K_t,

$$\delta\lambda_{t+1}\left(\alpha\left(\frac{\mathcal{N}_t}{K_{t-1}}\right)^{1-\alpha} + 1 - \delta_k\right) = \lambda_t \tag{A.33}$$

All curly functions are evaluated at the Ramsey plan $\{r_t, w_t\}_{t=0}^{\infty}$. In addition to the two FOCs, the implementability condition (37) itself, of course, also has to hold along the Ramsey plan.

Now consider the situation assumed in proposition 8. The Ramsey plan $\{r_t, w_t, K_t\}_{t=0}^{\infty}$ is assumed to converge to a tuple (r, w, K); and the multiplier λ_t is assumed to converge, too, to some value λ. For now, we assume $\lambda \neq 0$. By lemma 2, we know that this means that all the curly functions converge to their steady state values, evaluated at (r, w). In particular, the implementability condition (37),

$$G + K_t - (1-\delta_k) K_{t-1} + (1+r_t) \mathcal{A}_{t-1} \left(\{r_s, w_s\}_{s=0}^{\infty}\right) \\ = \mathcal{A}_t \left(\{r_s, w_s\}_{s=0}^{\infty}\right) + K_{t-1}^{\alpha} \mathcal{N}_t \left(\{r_s, w_s\}_{s=0}^{\infty}\right)^{1-\alpha} - w_t \mathcal{N}_t \left(\{r_s, w_s\}_{s=0}^{\infty}\right) \quad \text{(A.34)}$$

must converge, as $t \to \infty$, to the steady state government budget constraint

$$G + \delta_k K + r\mathcal{A}^{ss}(r, w) = K^{\alpha} \mathcal{N}^{ss}(r, w)^{1-\alpha} - w\mathcal{N}^{ss}(r, w)$$

which is exactly (39).

The RSS optimality condition (40) is derived just like in the proof of proposition 8, only that w^* is no longer equal to 1 here, but instead given by $w^* = (1-\alpha)\left(\frac{K}{\mathcal{N}^{ss}(r,w)}\right)^{\alpha}$. The modified golden rule (41) follows directly from (A.33) and $\lambda_t \to \lambda \neq 0$. This proves proposition 8.

B.12 Proof of proposition 9

Conditions (44) and (45) follow exactly as in the proof of proposition 5, just with a pre-tax wage

$$w^* = (1-\alpha)\left(\frac{K}{\mathcal{N}^{ss}(r, w)}\right)^{\alpha} \quad \text{(A.35)}$$

that may differ from 1.

To derive (46), substitute $\lambda_t / \lambda_{t-1} \to \eta$ into (A.33) to obtain

$$\delta\eta \left(\alpha \left(\frac{\mathcal{N}^{ss}(r, w)}{K}\right)^{1-\alpha} + 1 - \delta_k\right) = 1 \quad \text{(A.36)}$$

Substituting (A.36) into (A.35) yields (46).

B.13 Proof of proposition 10

The derivation of the immiseration-adjusted optimality condition (48) follows exactly as in the proof of proposition 6. The derivation of (49) follows that of (46) in the proof of proposition 9.

To derive (47), consider the implementability condition (37),

$$G + K_t - (1-\delta_k) K_{t-1} + (1+r_t) \mathcal{A}_{t-1} \left(\{r_s, w_s\}_{s=0}^{\infty}\right) \\ = \mathcal{A}_t \left(\{r_s, w_s\}_{s=0}^{\infty}\right) + K_{t-1}^{\alpha} \mathcal{N}_t \left(\{r_s, w_s\}_{s=0}^{\infty}\right)^{1-\alpha} - w_t \mathcal{N}_t \left(\{r_s, w_s\}_{s=0}^{\infty}\right) \quad \text{(A.37)}$$

De-trending the equation as in the proof of proposition 6, we have

$$G + K_t - (1-\delta_k) K_{t-1} + (1+\hat{r}_t) \gamma^t \mathcal{A}_{t-1} \left(\{\hat{r}_s, \hat{w}_s\}_{s=0}^{\infty}\right) \\ = \gamma^t \mathcal{A}_t \left(\{\hat{r}_s, \hat{w}_s\}_{s=0}^{\infty}\right) + K_{t-1}^{\alpha} \mathcal{N}_t \left(\{\hat{r}_s, \hat{w}_s\}_{s=0}^{\infty}\right)^{1-\alpha} - \hat{w}_t \gamma^t \mathcal{N}_t \left(\{\hat{r}_s, \hat{w}_s\}_{s=0}^{\infty}\right) \quad \text{(A.38)}$$

where, as before, $1+\hat{r}_t \equiv (1+r_t)/\gamma$ and $\hat{w}_t \equiv w_t/\gamma^t$. Taking limits $t \to \infty$, we arrive at

$$G = K^{\alpha} \mathcal{N}^{ss} (\hat{r}, \hat{w})^{1-\alpha} - \delta_k K$$

Substituting (A.35) into this equation, we obtain (47).

B.14 Proof of proposition 11

The Ramsey planning problem with lump-sum transfers is to maximize the objective (A.46) subject to the implementability constraint (A.47) as well as the non-negativity constraint for transfers $T_t \geq 0$. It is straightforward to see that the first-order condition trading off r_t and w_t is unchanged and given by (25) as in proposition 3, irrespective of whether the $T_t \geq 0$ binds or not. It is also straightforward to see that the steady state government budget constraint (A.50) has to hold.

The first order condition w.r.t. r_s is given by

$$\sum_{h=-s}^{\infty} \delta^h \frac{\partial \mathcal{U}_{s+h}}{\partial T_s} + \sum_{h=-s}^{\infty} \delta^h \lambda_{s+h} \left(\frac{\partial \mathcal{A}_{s+h}}{\partial T_s} + (1-w_{s+h}) \frac{\partial \mathcal{N}_{s+h}}{\partial T_s} - (1+r_{s+h}) \frac{\partial \mathcal{A}_{s+h-1}}{\partial T_s} \right) - \lambda_s + \mu_s = 0 \quad \text{(A.39)}$$

where $\mu_s \geq 0$ is the shadow value of relaxing the $T_s \geq 0$ constraint. The first order condition w.r.t. r_s is (A.10) as before,

$$\sum_{h=-s}^{\infty} \delta^h \frac{\partial \mathcal{U}_{s+h}}{\partial r_s} + \sum_{h=-s}^{\infty} \delta^h \lambda_{s+h} \left(\frac{\partial \mathcal{A}_{s+h}}{\partial r_s} + (1-w_{s+h}) \frac{\partial \mathcal{N}_{s+h}}{\partial r_s} - (1+r_{s+h}) \frac{\partial \mathcal{A}_{s+h-1}}{\partial r_s} \right) - \lambda_s \mathcal{A}_{s-1} = 0 \quad \text{(A.40)}$$

Assuming that the economy converges to a Ramsey steady state characterized by (r, w, T) with converging multiplier $\lambda_t \to \lambda \neq 0$, (A.40) converges to (A.12) in the limit,

$$\epsilon^{U,r}(\delta) + \lambda \left(A\epsilon^{A,r}(\delta) + (1-w) N\epsilon^{N,r}(\delta) - \delta (1+r) A\epsilon^{A,r}(\delta) \right) - \lambda A = 0 \quad \text{(A.41)}$$

Given definition (A.49), the FOC w.r.t. T (A.39) becomes

$$\epsilon^{U,T}(\delta) + \lambda \left(A\epsilon^{A,T}(\delta) + (1-w)\, N\epsilon^{N,T}(\delta) - \delta\,(1+r)\, A\epsilon^{A,T}(\delta) \right) - \lambda N w = 0 \tag{A.42}$$

if T_s is positive in the limit, that is, $\liminf_{s\to\infty} T_s > 0$, and thus $\lim_{s\to\infty} \mu_s = 0$. Otherwise,

$$\epsilon^{U,T}(\delta) + \lambda \left(A\epsilon^{A,T}(\delta) + (1-w)\, N\epsilon^{N,T}(\delta) - \delta\,(1+r)\, A\epsilon^{A,T}(\delta) \right) - \lambda N w \leq 0$$

Substituting (A.41) into (A.42), we then find

$$\frac{\epsilon^{U,T}(\delta)}{\epsilon^{U,r}(\delta)} - \frac{(1-\delta\,(1+r))\,\ell\epsilon^{A,T}(\delta) + \frac{1-w}{w}\epsilon^{N,T}(\delta) - 1}{(1-\delta\,(1+r))\,\ell\epsilon^{A,r}(\delta) + \frac{1-w}{w}\epsilon^{N,r}(\delta) - \ell} = 0$$

if T_s is positive in the limit, that is, $\liminf_{s\to\infty} T_s > 0$. Else, the equation holds with weak inequality "$\leq$". Thus, if we know that the FOC holds with strict inequality "$<$", then it must be that the Ramsey steady state has zero transfers in the limit, $T = 0$.

B.15 Proof of proposition 12

We first show that the discounted elasticities w.r.t. T_s are well-defined, even in the immiseration limit. Consider the following modified household problem:

$$\max \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_{it}, n_{it}) \right]$$

subject to

$$c_{it} + a_{it} = (1+r_t)\, a_{it-1} + x_t w_t N_t + w_t e_{it} n_{it}$$

where $x_t \geq 0$ is an exogenous sequence representing a lump-sum transfer that is scaled by $w_t N_t$. The curly functions characterizing aggregate household behavior $\mathcal{U}, \mathcal{A}, \mathcal{N}$, are now functions of r_t, w_t as well as x_t and N_t.

Following the exact same steps as in the proof of lemma 5, we can show that with log-separable preferences,

$$\mathcal{U}_t\left(\{r_s, w_s, x_s, N_s\}\right) = \mathcal{U}_t\left(\{\hat{r}_s, \hat{w}_s, x_s, N_s\}\right) \tag{A.43}$$

$$\mathcal{A}_t\left(\{r_s, w_s, x_s, N_s\}\right) = \gamma^t \mathcal{A}_t\left(\{\hat{r}_s, \hat{w}_s, x_s, N_s\}\right) \tag{A.44}$$

$$\mathcal{N}_t\left(\{r_s, w_s, x_s, N_s\}\right) = \mathcal{N}_t\left(\{\hat{r}_s, \hat{w}_s, x_s, N_s\}\right) \tag{A.45}$$

where, as before,

$$1 + \hat{r}_s \equiv \frac{1+r_s}{\gamma} \qquad \hat{w}_s \equiv w_s \gamma^{-s}$$

This implies that derivatives w.r.t. x_s can be transformed as follows

$$\frac{\partial \mathcal{U}_{s+h}\left(\{r_s, w_s, x_s, N_s\}\right)}{\partial x_s} = \frac{\partial \mathcal{U}_{s+h}\left(\{\hat{r}_s, \hat{w}_s, x_s, N_s\}\right)}{\partial x_s}$$

Once we evaluate this equation around $x_s = 0$ and exchange variable x_t for the lump-sum transfer $T_t = x_t w_t N_t$, we obtain that the normalization in (A.49) allows the discounted derivatives to be scale invariant,

$$\frac{\partial \mathcal{U}_{s+h}\left(\{r_s, w_s\}\right)}{\partial T_s} w_s N_s = \frac{\partial \mathcal{U}_{s+h}\left(\{\hat{r}_s, \hat{w}_s\}\right)}{\partial T_s} \hat{w}_s N_s$$

We next apply this logic to derive (A.51). Under the assumptions of proposition 6, $w_s \to 0$ with some decay factor $\gamma \in (0,1)$, and λ_s diverges with some factor $\eta > 1$. Consider now the first-order condition for transfers (A.39),

$$\sum_{h=-s}^{\infty} \delta^h \frac{\partial \mathcal{U}_{s+h}}{\partial T_s} + \sum_{h=-s}^{\infty} \delta^h \lambda_{s+h} \left(\frac{\partial \mathcal{A}_{s+h}}{\partial T_s} + (1 - w_{s+h}) \frac{\partial \mathcal{N}_{s+h}}{\partial T_s} - (1 + r_{s+h}) \frac{\partial \mathcal{A}_{s+h-1}}{\partial T_s} \right) - \lambda_s \leq 0$$

Divide this equation by λ_s and multiply with $w_s N_s$,

$$\frac{1}{\lambda_s} \sum_{h=-s}^{\infty} \delta^h \frac{\partial \mathcal{U}_{s+h}}{\partial T_s} w_s N_s + \sum_{h=-s}^{\infty} \delta^h \frac{\lambda_{s+h}}{\lambda_s} \left(\frac{\partial \mathcal{A}_{s+h}}{\partial T_s} w_s N_s + (1 - w_{s+h}) \frac{\partial \mathcal{N}_{s+h}}{\partial T_s} w_s N_s - (1 + r_{s+h}) \frac{\partial \mathcal{A}_{s+h-1}}{\partial T_s} w_s N_s \right) - w_s N_s \leq$$

We now express the first order condition in terms of scale-free variables $\hat{r}_t$, $\hat{w}_t$, and denote curly functions evaluated around those with a hat. Then,

$$\frac{1}{\lambda_s} \sum_{h=-s}^{\infty} \delta^h \frac{\partial \hat{\mathcal{U}}_{s+h}}{\partial T_s} \hat{w}_s N_s + \sum_{h=-s}^{\infty} \delta^h \frac{\lambda_{s+h}}{\lambda_s} \left(A_{s+h} \frac{\partial \log \hat{\mathcal{A}}_{s+h}}{\partial T_s} \hat{w}_s N_s + \left(1 - \gamma^{s+h} \hat{w}_{s+h}\right) N_{s+h} \frac{\partial \log \hat{\mathcal{N}}_{s+h}}{\partial T_s} \hat{w}_s N_s - (1 + r_{s+h}) A \right.$$

Taking limits $s \to \infty$, all terms converge to zero, except the labor derivative. We thus find

$$\lim_{s \to \infty} \sum_{h=-s}^{\infty} \delta^h \frac{\lambda_{s+h}}{\lambda_s} N_{s+h} \frac{\partial \log \hat{\mathcal{N}}_{s+h}}{\partial T_s} \hat{w}_s N_s \leq 0$$

or in other words,

$$\epsilon^{N,T}(\delta\eta) \leq 0$$

This proves the proposition.

B.16 Optimality conditions with a poverty state

In this section we analyze an economy with two kinds of agents. A mass $1 - \mu$ of heterogeneous agents maximize utility (1) with KPR flow utility (10) subject to (2) and (3). We re-normalize their productivity such that $\mathbb{E}e_{it} = \frac{1-\xi}{1-\mu}$. A mass μ of agents has no income risk, which effectively means they will be hand-to-mouth with a binding borrowing constraint. We assume their productivity is given by ξ/μ.

We continue to use $\mathcal{A}, \mathcal{N}, \mathcal{C}, \mathcal{U}$ for the sequence-space functions characterizing behavior of a continuum of heterogeneous agents with $\mathbb{E}e_{it} = 1$. Note that balanced growth preferences imply that effective hours worked of hand-to-mouth agents are constant, at some level N^h.

Combining both groups of agents, we find consolidated sequence-space functions

$$\overline{\mathcal{A}}_t\left(\{r_s, w_s\}\right) = (1-\xi)\,\mathcal{A}_t\left(\{r_s, w_s\}\right)$$

$$\overline{\mathcal{C}}_t\left(\{r_s, w_s\}\right) = (1-\xi)\,\mathcal{C}_t\left(\{r_s, w_s\}\right) + \xi w_t N^h$$

$$\overline{\mathcal{N}}_t\left(\{r_s, w_s\}\right) = (1-\xi)\,\mathcal{N}_t\left(\{r_s, w_s\}\right) + \xi N^h$$

$$\overline{\mathcal{U}}_t\left(\{r_s, w_s\}\right) = (1-\mu)\left(\frac{1-\xi}{1-\mu}\right)^{1-\sigma}\mathcal{U}_t\left(\{r_s, w_s\}\right) + \mu\left(\frac{\xi}{\mu}\right)^{1-\sigma} u\left(w_t N^h, N^h\right)$$

These consolidated functions can be used to obtain discounted elasticities of the entire consolidated household side of the economy. This allows to evaluate the RSS optimality conditions in section 3.

In the following, we focus on the special case used in section 5.2, with log-separable preferences (11), that is, $\sigma = 1$. Further, we assume productivity of hand-to-mouth households is exceedingly small, $\xi \to 0$. This lets us interpret the hand-to-mouth households as being in a "poverty state".[A-2]

In this case, the consolidated sequence-space functions simply boil down to

$$\overline{\mathcal{A}}_t = \mathcal{A}_t \qquad \overline{\mathcal{C}}_t = \mathcal{C}_t \qquad \overline{\mathcal{N}}_t = \overline{\mathcal{N}}$$

$$\overline{\mathcal{U}}_t\left(\{r_s, w_s\}\right) = (1-\mu)\,\mathcal{U}_t\left(\{r_s, w_s\}\right) + \mu \log w_t + const$$

where the constant is given by $\log N^h - v(N^h)$. Discounted elasticities of $\mathcal{A}, \mathcal{C}, \mathcal{N}$ are thus unchanged relative to the model without poverty state ($\mu = \xi = 0$). Discounted derivatives of utility are given by

$$\epsilon^{\overline{U},r} = (1-\mu)\,\epsilon^{U,r} \qquad \epsilon^{\overline{U},w} = (1-\mu)\,\epsilon^{U,w} + \mu$$

The effective marginal rate of substitution between an interest rate and a wage increase, as defined in (21), is then given by

$$\overline{\text{mrs}} = \frac{\epsilon^{\overline{U},w}}{\epsilon^{\overline{U},r}} = \frac{\epsilon^{U,w} + \frac{\mu}{1-\mu}}{\epsilon^{U,r}}$$

This is, in fact, the only object that changes in the RSS optimality conditions, such as (25).

[A-2] An alternative equivalent way to arrive at this limit is to introduce an additional low-productivity state into the original Markov chain for e_{it}, and then assume vanishingly small transition probabilities in and out of that state.

C Algorithm to compute discounted elasticities

We use the discretized household side described by equations (**??**) – (**??**) and ask how we can best compute the discounted derivative of outcome y_t with respect to input x_t,

$$\epsilon^{y,x} \equiv \lim_{s\to\infty} \sum_{h=-s}^{\infty} \delta^h \frac{\partial y_{s+h}}{\partial x_s}$$

This approach can be used to compute discounted derivatives or elasticities of any household outputs w.r.t. household inputs. We use the notation of appendix **??** throughout this section. In particular, we continue to denote by $\mathbf{F}$ the fake-news matrix, defined as

$$F_{t,0} = \begin{cases} \mathbf{Y}_x'\mathbf{D}_{ss} & t=0 \\ \mathbf{Y}_{ss}'\left(\Lambda_{ss}'\right)^{t-1}\Lambda D_x & t>0 \end{cases}$$

and for $s>0$ as

$$F_{t,s} = \begin{cases} \mathbf{v}_x'\left(\mathbf{v}_{\mathbf{v}}'\right)^{s-1}\mathbf{Y}_{\mathbf{v}}'\mathbf{D}_{ss} & t=0 \\ \mathbf{Y}_{ss}'\left(\Lambda_{ss}'\right)^{t-1}\Lambda D_{\mathbf{v}}\left(\mathbf{v}_{\mathbf{v}}\right)^{s-1}\mathbf{v}_x & t>0 \end{cases}$$

Per our assumption **??**, recall that the derivatives $\frac{\partial y_{s+h}}{\partial x_s}$ for large s converge to j_h, defined in (**??**),

$$j_h \equiv \sum_{v=\max\{0,-h\}}^{\infty} F_{h+v,v}$$

That is,

$$\epsilon^{y,x} \equiv \sum_{h=-\infty}^{\infty} \delta^h j_h$$

Substituting in the expressions for the elements of $\mathbf{F}$ into this expression, we find, after some lengthy algebra,

$$\epsilon^{y,x} = \mathbf{Y}_x'\mathbf{D}_{ss} + \mathbf{Y}_{ss}'\left(\mathbf{I}-\delta\Lambda_{ss}'\right)^{-1}\delta\Lambda D_x + \left(\mathbf{D}_{ss}'\mathbf{Y}_{\mathbf{v}} + \mathbf{Y}_{ss}'\left(\mathbf{I}-\delta\Lambda_{ss}'\right)^{-1}\delta\Lambda D_{\mathbf{v}}\right)\left(\mathbf{I}-\delta^{-1}\mathbf{v}_{\mathbf{v}}\right)^{-1}\delta^{-1}\mathbf{v}_x$$

We evaluate this expression in three simple steps. We start by introducing two auxiliary objects.

Step 1: Backward iteration. Shocking x directly affects the policy by $\mathbf{v}_x$. Iterating backwards we can evaluate the discounted derivative of the policy function,

$$\mathbf{s}^{pol} \equiv \left(\mathbf{I}-\delta^{-1}\mathbf{v}_{\mathbf{v}}\right)^{-1}\delta^{-1}\mathbf{v}_x$$

recursively via a sequence $\mathbf{s}_n^{pol}$,

$$\mathbf{s}_{n+1}^{pol} \equiv \delta^{-1}\mathbf{v}_x + \delta^{-1}\mathbf{v}_{\mathbf{v}}\mathbf{s}_n^{pol}$$

starting from $\mathbf{s}_0^{pol} = 0$, with the property that $\mathbf{s}_n^{pol} \to \mathbf{s}^{pol}$. This recursion has the computational cost of a single backward iteration.

Step 2: Expectations iteration. We next compute the discounted expectations vector, defined as

$$\mathbf{s}^{exp} \equiv (\mathbf{I} - \delta\Lambda_{ss})^{-1}\,\mathbf{Y}_{ss}$$

For each idiosyncratic state, this vector captures the expected discounted sum of future outputs y. Just like before, we compute this object recursively via a sequence $\mathbf{s}_n^{exp}$,

$$\mathbf{s}_{n+1}^{exp} \equiv \mathbf{Y}_{ss} + \delta\Lambda_{ss}\mathbf{s}_n^{exp}$$

starting from $\mathbf{s}_0^{exp} = 0$. This recursion ensures that $\mathbf{s}_n^{exp} \to \mathbf{s}^{exp}$ and has the computational cost of a single forward iteration.

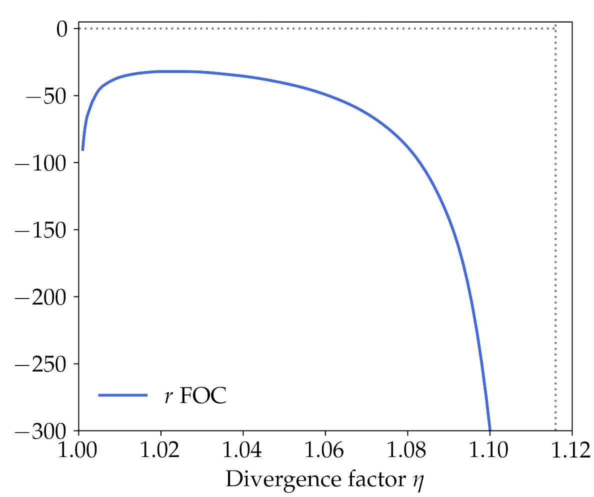
Step 3: Evaluating $\epsilon^{y,x}$. With $\mathbf{s}^{pol}$ and $\mathbf{s}^{exp}$, we can rewrite the expression for $\epsilon^{y,x}$ as

$$\epsilon^{y,x} = \mathbf{Y}_x'\mathbf{D}_{ss} + \mathbf{s}^{exp\prime}\delta\Lambda D_x + \left(\mathbf{D}_{ss}'\mathbf{Y}_{\mathbf{v}} + \mathbf{s}^{exp\prime}\delta\Lambda D_{\mathbf{v}}\right)\mathbf{s}^{pol}$$

which is straightforward to evaluate. This is how we compute discounted elasticities.

D Additional figures

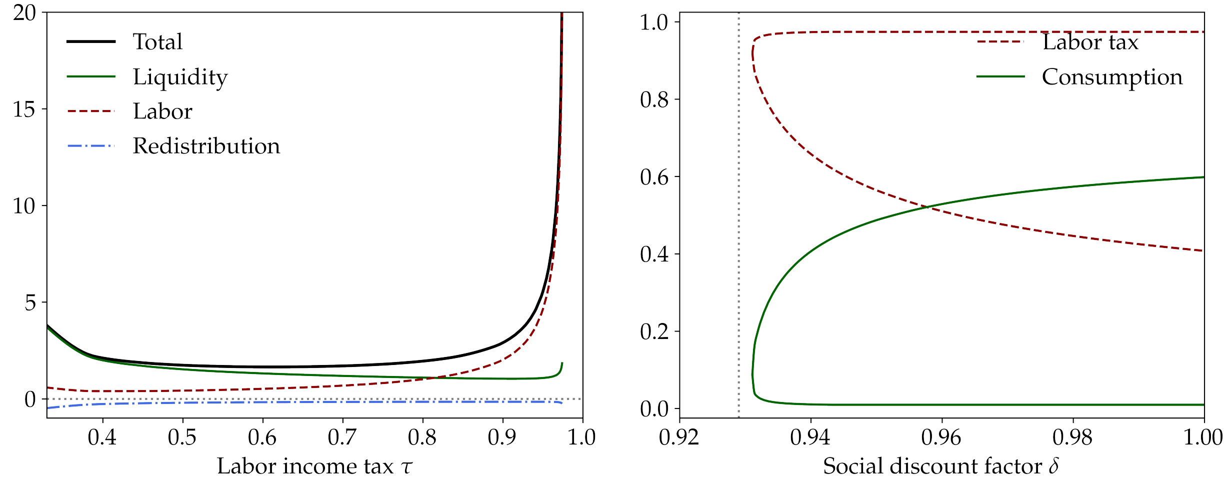
Figure D.1: First order conditions with diverging multipliers



Note: This figure displays the diverging multiplier first-order condition for r as the divergence factor η is varied, evaluated at the market-clearing values of r, τ such that the τ first-order condition equals zero.

Figure D.2: RSS for additively separable preferences with EIS > 1, high G

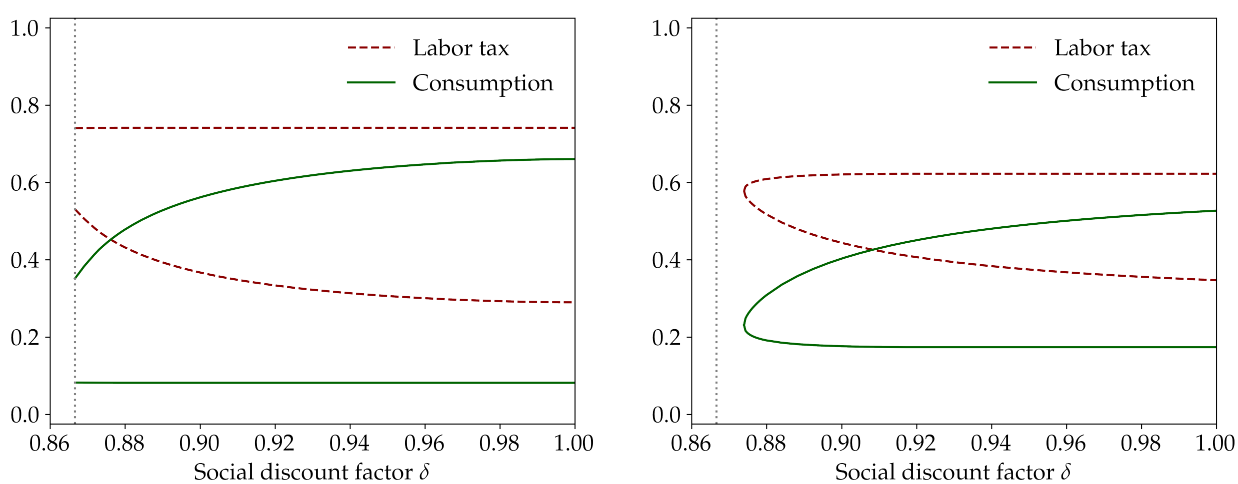
(a) Net liquidity benefit | **(b)** Varying the social discount factor



Note: Panel (a) displays the three terms of the interior Ramsey steady state optimality condition (stated in proposition 3) and their sum for a modified calibration with EIS $\sigma^{-1} = 2$ and a government spending-to-GDP ratio of $G/Y = 0.31$. Panel (b) displays the interior Ramsey steady state values of labor taxes and aggregate consumption attained as the social discount factor δ is varied from $\delta = \beta$ (grey-dotted vertical line) to $\delta = 1$.

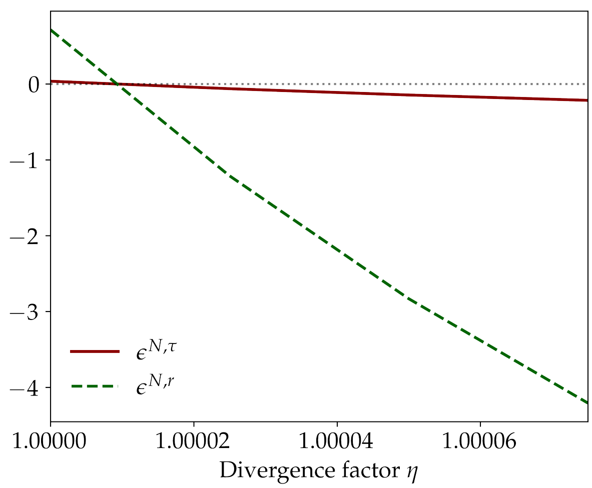
Figure D.3: Varying the social discount factor with GHH preferences

(a) Low government spending G | **(b)** High government spending G



Note: The figures display the interior Ramsey steady state values of labor taxes and aggregate consumption attained as the social discount factor δ is varied from $\delta = \beta$ (grey-dotted vertical line) to $\delta = 1$. Government spending-to-GDP ratio is $G/Y = 0.25$ in panel (a) and 0.31 in panel (b).

Figure D.4: Looking for immiseration in the economy with capital



Note: This figure displays labor elasticities with respect to labor income taxes and interest rates for the optimal Ramsey plan whose Lagrange multipliers diverge at exponential rate η for a modified calibration including capital, where the capital-output ratio $K/Y = 3$, depreciation rate $\delta_k = 0.05$, and capital share $\alpha = 0.3$.

E Alternative tax instruments

So far, we have allowed for two tax instruments: labor and capital taxes. As usual, allowing for consumption taxes would not change anything as a consumption tax in our economy can be replicated using capital and labor income taxes. In this section, we consider two alternative additional tax instruments, a lump-sum transfer and time-varying tax progressivity.

E.1 Lump-sum transfers

Denote by $T_t \geq 0$ an non-negative lump-sum transfer the planner has at its disposal. The path of transfers $\{T_t\}_{t=0}^{\infty}$ now enters the sequence-space functions introduced in (2.1). In particular, the planning problem now becomes

$$\sum_{t=0}^{\infty} \delta^t \mathcal{U}_t \left(\{r_s, w_s, T_s\}\right) \tag{A.46}$$

subject to a modified implementability condition

$$G + T_t + (1 + r_t)\, \mathcal{A}_{t-1}\left(\{r_s, w_s, T_s\}_{s=0}^{\infty}\right) = \mathcal{A}_t\left(\{r_s, w_s, T_s\}_{s=0}^{\infty}\right) + (1 - w_t)\, \mathcal{N}_t\left(\{r_s, w_s, T_s\}_{s=0}^{\infty}\right) \tag{A.47}$$

as well as the non-negativity condition for transfers, $T_t \geq 0$. Under the assumption of converging multipliers λ_t on the implementability condition (A.47) and positive transfers $T_t > 0$ in the Ramsey steady state, we can derive an additional first-order optimality condition,

$$\frac{\epsilon^{U,T}(\delta)}{\epsilon^{U,r}(\delta)} - \frac{(1-\delta(1+r))\,\ell\epsilon^{A,T}(\delta) + \frac{1-w}{w}\epsilon^{N,T}(\delta) - 1}{(1-\delta(1+r))\,\ell\epsilon^{A,r}(\delta) + \frac{1-w}{w}\epsilon^{N,r}(\delta) - \ell} = 0 \tag{A.48}$$

This condition is analogous to (25), except that all tax (and wage) elasticities are replaced by elasticities with respect to the lump-sum transfer. We define the elasticities with respect to T_t as

$$\epsilon^{N,T}(\delta) \equiv \lim_{s\to\infty} \sum_{h=-s}^{\infty} \delta^h \frac{\partial \log \mathcal{N}_{s+h}}{\partial T_s} N_s w_s \tag{A.49}$$

where the normalization happens with respect to after-tax income $N_s w_s$, not T_s, as the transfer may be zero. All other elasticities in (A.48) are defined analogously. We summarize optimality in the following result.

Proposition 11. *If the two prices and the transfer (r, w, T) are part of a Ramsey steady state of a Ramsey plan with converging Lagrange multipliers λ_t in the economy with lump-sum transfers, then the steady state government budget constraint has to hold,*

$$G + T + r\mathcal{A}^{ss}(r, w) = (1-w)\,\mathcal{N}^{ss}(r, w) \tag{A.50}$$

as well as (25); if $T > 0$, (A.48) has to hold with equality; if (A.48) holds with strict inequality "$<$", $T = 0$.

Proposition 11 essentially provides a Kuhn-Tucker condition for T which gives us a way to investigate whether lump-sum transfers are zero in the Ramsey steady state. In light of our results on the prevalence of immiseration in section 4, however, we'd like a version of proposition 11 that can be applied along a transition to immiseration. This is what the following proposition delivers.

Proposition 12. *Assume $u(c, n)$ is log-separable, as in (11). Let $\{r_t, w_t\}$ satisfy the requirements and conditions (29) and (30) in proposition 6. If, in addition, transfers T_t converge to zero relative to wages w_t, we must have that*

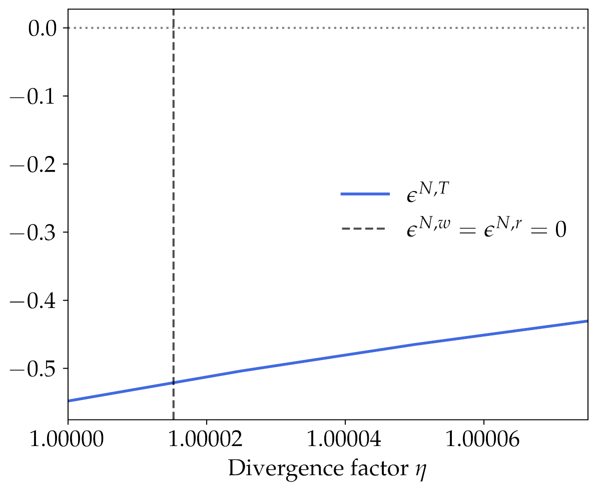
$$\epsilon^{N,T}(\delta\eta) \leq 0 \tag{A.51}$$

evaluated at the de-trended steady state described in proposition 6.

Condition (A.51) is a necessary optimality condition if zero transfers are optimal as the economy tends toward immiseration. In fact, while we cannot show this formally so far, we suspect that if (A.51) holds with strict inequality, zero transfers are locally optimal.

In figure E.1, we plot (A.51) as a function of the factor $\eta \geq 1$ with which the Lagrange multiplier diverges (see also figure 6). As usual, we set $\delta = \beta$. We see that (A.51) holds with strict inequality at the η associated with immiseration, suggesting that lump-sum transfers are not optimal to use as the economy tends to immiseration. The interpretation of this result is that if reducing lump-sum transfers raises labor supply, $\epsilon^{N,T} < 0$, then the planner has no interest in setting positive transfers.

Figure E.1: The net benefit of using lump-sum transfers during immiseration



Note: This figure displays the labor elasticity with respect to a lump-sum transfer for the optimal Ramsey plan whose Lagrange multipliers diverge at exponential rate η. $\epsilon^{N,w} = \epsilon^{N,r} = 0$ (grey dashed) indicates the rate η at the optimum.

E.2 Time-varying tax progressivity

Lump-sum transfers are one way of making taxes more progressive. A second way, which has recently become very popular, uses the Heathcote, Storesletten and Violante (2017) power retention function, which replaces the date-t budget constraint of each household i, (2), with

$$c_{it} + a_{it} = (1 + r_t)\, a_{it-1} + \eta_t \left(w^* e_{it} n_{it}\right)^{1-\varphi_t} \tag{A.52}$$

Here, the government controls the slope η_t as well as the tax progressivity parameter $\varphi_t \in [0, 1)$. To remain close to our previous analysis, we define

$$w_t \equiv \eta_t \left(w^*\right)^{1-\varphi_t} \mathbb{E}\left[e_{it}^{1-\varphi_t}\right] (1 - \varphi_t)$$

so that the budget constraint (A.52) becomes

$$c_{it} + a_{it} = (1 + r_t)\, a_{it-1} + w_t \frac{1}{1 - \varphi_t} \frac{e_{it}^{1-\varphi_t}}{\mathbb{E}\left[e_{it}^{1-\varphi_t}\right]} n_{it}^{1-\varphi_t}$$

Formulated in this way, the return to an additional marginal hour worked is given by

$$w_t \frac{e_{it}^{1-\varphi_t}}{\mathbb{E}\left[e_{it}^{1-\varphi_t}\right]} n_t^{-\varphi_t}$$

This shows how greater tax progressivity effectively lowers the dispersion in tax-adjusted productivities via the term $e_{it}^{1-\varphi_t}/\mathbb{E}\left[e_{it}^{1-\varphi_t}\right]$, and reduces the after-tax return on an hour worked for those working long hours.

To allow for φ_t to be chosen optimally, we include the path of tax progressivities $\{\varphi_t\}$ as an additional argument in the sequence space functions introduced in section 2.1. We also define a new one—labor tax revenue—as

$$\mathcal{T}_t\left(\{r_s, w_s, \varphi_s\}_{s=0}^{\infty}\right) \equiv \int w^* e_{it} n_{it} di - \int w_t \frac{1}{1-\varphi_t} \frac{e_{it}^{1-\varphi_t}}{\mathbb{E}\left[e_{it}^{1-\varphi_t}\right]} n_{it}^{1-\varphi_t} di$$

where, as before, $w^* = 1$. We define elasticities with respect to φ as

$$\epsilon^{N,\varphi}(\delta) \equiv \lim_{s\to\infty} \sum_{h=-s}^{\infty} \delta^h \frac{\partial \log \mathcal{N}_{s+h}}{\partial \varphi_s}$$

and similarly for the other sequence-space functions.

The planning problem then is to maximize

$$\sum_{t=0}^{\infty} \delta^t \mathcal{U}_t\left(\{r_s, w_s, \varphi_s\}\right)$$

subject to

$$G + (1+r_t)\,\mathcal{A}_{t-1}\left(\{r_s, w_s, \varphi_s\}_{s=0}^{\infty}\right) = \mathcal{A}_t\left(\{r_s, w_s, \varphi_s\}_{s=0}^{\infty}\right) + \mathcal{T}_t\left(\{r_s, w_s, \varphi_s\}_{s=0}^{\infty}\right)$$

Just like before, we can characterize a set of first order conditions that describe the optimal choice of the progressivity parameter φ in a Ramsey steady state. The conditions are analogous to the ones in the previous subsection. When the Lagrange multiplier λ_t converges, we have the additional first order condition

$$(1-\delta(1+r))\,\ell\left(\frac{\epsilon^{U,\varphi}}{\epsilon^{U,r}}\epsilon^{A,r} - \epsilon^{A,\varphi}\right) - \frac{1-w}{w}\left(\epsilon^{N,\varphi} - \frac{\epsilon^{U,\varphi}}{\epsilon^{U,r}}\epsilon^{N,r}\right) - \left(\ell\frac{\epsilon^{U,\varphi}}{\epsilon^{U,r}} - 1\right) = 0$$

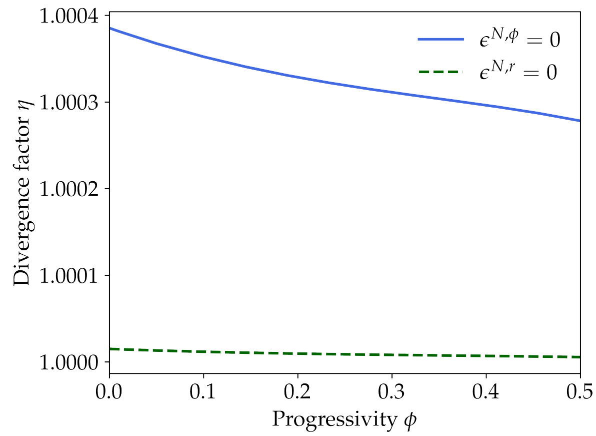
which, together with (24) and (25) pins down the tuple r, w, φ that defines the Ramsey steady state. If the economy, instead, tends to immiseration with some interior optimal progressivity $\varphi \in (0,1)$, the condition

$$\epsilon^{N,\varphi}(\delta\eta) = 0 \tag{A.53}$$

has to hold, in addition to (29) and (30). If instead φ is at the lower boundary, $\varphi = 0$, then we need to have $\epsilon^{N,\varphi}(\delta\eta) \leq 0$, at the divergence factor η that solves $\epsilon^{N,r}(\delta\eta) = 0$.

Figure E.2 plots the two conditions (A.53) and (30) as loci in the space of φ, η combinations. $\epsilon^{N,\varphi} < 0$ below the blue locus, indicating that, indeed, $\varphi = 0$. Progressive taxes are not optimal as

Figure E.2: Optimal tax progressivity



Note: This figure displays the points along the labor tax progressivity ϕ and divergence rate η plane consistent with the optimal Ramsey plan with diverging Lagrange multipliers.

the economy tends to immiseration.

F Additional model extensions

F.1 Endogenous government spending

So far, we have considered the case of exogenous government spending $G > 0$. In this section, we consider the case of endogenous government spending G. There are two canonical approaches to endogenizing G. The first approach assumes that government spending is a fixed fraction of output; this gives rise to an incentive for the planner to distort output downward solely to reduce wasteful government spending. The approach is to include G in the utility function of households and assume that the planner sets G optimally. This is what we explore next.

Households' per-period utility is now given by $u(c, n) + U(G)$. The Ramsey problem is then to maximize

$$\sum_{t=0}^{\infty} \delta^t \left(\mathcal{U}_t \left(\{ r_s, w_s \} \right) + U(G_t) \right)$$

subject to the same implementability condition (16) as before. It is straightforward to see that the endogenous choice of G_t gives rise to a new first-order condition,

$$U'(G_t) = \lambda_t$$

With this condition at hand, we can restate the necessary RSS optimality conditions in proposition 3. The government budget constraint (24) and the optimality condition (25) still need to hold. In addition, as (A.14) in the proof of proposition 3 reveals, the terminal value of the Lagrange multiplier is given by

$$\lambda = \frac{A^{-1}\ell\epsilon^{U,r}(\delta)}{\ell - (1-\delta(1+r))\,\ell\epsilon^{A,r}(\delta) - \frac{1-w}{w}\epsilon^{N,r}(\delta)}$$

Thus, the third condition is then

$$U'(G) = \frac{A^{-1}\ell\epsilon^{U,r}(\delta)}{\ell - (1-\delta(1+r))\,\ell\epsilon^{A,r}(\delta) - \frac{1-w}{w}\epsilon^{N,r}(\delta)}$$

The three conditions pin down the tuple (r, w, G).

Observe that when household utility u is log-separable, then by our formula for λ, (A.23), in the proof of proposition 4, we have that

$$\lambda = \frac{1}{rA + wN} = \frac{1}{C}$$

Thus, in this case, we simply have

$$U'(G) = \frac{1}{C}$$

which is the classic Samuelson (1954) condition for optimal public good provision.

F.2 Fixed lump-sum transfer

In this section, we describe a version of the Ramsey problem in which the government is obligated to pay a fixed exogenous transfer $T > 0$ to households. In this case, the planner maximizes (20) subject to

$$G + T + (1+r_t)\,\mathcal{A}_{t-1}\left(\{r_s, w_s\}_{s=0}^{\infty}\right) = \mathcal{A}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right) + (1-w_t)\,\mathcal{N}_t\left(\{r_s, w_s\}_{s=0}^{\infty}\right)$$

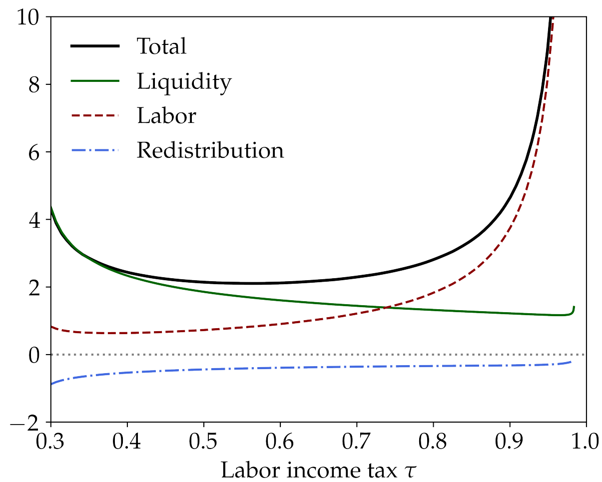
where now the sequence-space functions $\mathcal{A}, \mathcal{N}, \mathcal{U}$ are all derived from a consumption-saving model in which agents receive a lump-sum transfer $T > 0$. The RSS optimality condition in this economy is unchanged and given by (25); only the steady state government budget constraint now needs to include the transfer T as well.

Figure F.1 shows the three terms of the RSS optimality condition in our baseline calibration, with T fixed at 10% of initial GDP. As one can see, an interior RSS with converging multiplier does not exist in this economy either. We suspect, however, the one with a diverging multiplier (as in proposition 5) does exist.

F.3 Open economy

In this section we ask to what extent our insights also apply in a large open economy, such as the United States. To do so, we sketch a model of a world economy consisting of two nations, the home country, with a mass 1 of agents, characterized by the sequence space functions $\mathcal{A}, \mathcal{C}, \mathcal{N}, \mathcal{U}$; as

Figure F.1: RSS optimality condition in economy with fixed lump-sum transfer



Note: This figure displays the three terms of the interior Ramsey steady state optimality condition (stated in proposition 4) with a fixed lump-sum transfer T equal to 10% of initial GDP. Households have log-separable preferences and a zero-borrowing constraint, and the planner shares the households' preferences, $\delta = \beta$.

well as the rest of the world (ROW) with a population $\mu > 0$ of heterogeneous agents. To simplify the analysis, we assume the ROW has no taxes, no government spending, and no government debt itself. Moreover, production works with the same production function, producing the same good, pinning down the ROW real wage at 1. Finally, we assume that all the household primitives (the process for e_{it}, β, and the utility function $u(c,n)$) are identical in the ROW and in the home economy. Thus, the average behavior of ROW households is pinned down by the same sequence space functions $\mathcal{A}, \mathcal{C}, \mathcal{N}, \mathcal{U}$. We focus on the case where u is balanced growth compatible, of the form (10).

In this world economy, worldwide asset demand is given by

$$\overline{\mathcal{A}}_t\left(\{r_s, w_s\}\right) \equiv \underbrace{\mathcal{A}_t\left(\{r_s, w_s\}\right)}_{\text{asset demand of home households}} + \underbrace{\mu\mathcal{A}_t\left(\{r_s, 1\}\right)}_{\text{asset demand of ROW households}}$$

as both countries face the same interest rates $\{r_t\}$, but only home households are subject to labor income taxes, implying $w_t = 1 - \tau_t$.

Around some steady state r, w, the discounted elasticity of $\overline{\mathcal{A}}$ w.r.t. r is given by

$$\epsilon^{\overline{A},r}(\delta) = \frac{\mathcal{A}^{ss}(r,w)}{\mathcal{A}^{ss}(r,w) + \mu\mathcal{A}^{ss}(r,1)}\epsilon^{A,r}(\delta) + \frac{\mu\mathcal{A}^{ss}(r,1)}{\mathcal{A}^{ss}(r,w) + \mu\mathcal{A}^{ss}(r,1)}\epsilon^{A,r}(\delta)$$

Due to balanced growth compatibility, we have $\mathcal{A}^{ss}(r,w) = \mathcal{A}^{ss}(r,1)w$. Thus,

$$\epsilon^{\overline{A},r}(\delta) = \frac{w}{w+\mu}\epsilon^{A,r}(\delta) + \frac{\mu}{w+\mu}\epsilon^{A,r}(\delta) = \epsilon^{A,r}(\delta) \tag{A.54}$$

In other words, the elasticity of worldwide asset demand to interest rate changes is unchanged. Repeating the same with elasticities to wage changes, we find that it is instead lower,

$$\epsilon^{\overline{A},w}(\delta) = \frac{\mathcal{A}^{ss}(r,w)}{\mathcal{A}^{ss}(r,w) + \mu \mathcal{A}^{ss}(r,1)} \epsilon^{A,w}(\delta) = \frac{1}{1 + \mu / w} \epsilon^{A,w}(\delta) \tag{A.55}$$

which is intuitive as only home households' asset demand is responsive to wages. We define the following ratio of worldwide liquidity to domestic after-tax income,

$$\overline{\ell} \equiv \frac{\mathcal{A}^{ss}(r,w) + \mu \mathcal{A}^{ss}(r,1)}{w \mathcal{N}^{ss}(r,w)} = \left(1 + \frac{\mu}{w}\right) \ell \tag{A.56}$$

With $\overline{\mathcal{A}}_t$ at hand, the associated (government) implementability condition is given by

$$G + (1 + r_t) \overline{\mathcal{A}}_{t-1} \left(\{r_s, w_s\}_{s=0}^{\infty}\right) = \overline{\mathcal{A}}_t \left(\{r_s, w_s\}_{s=0}^{\infty}\right) + (1 - w_t) \mathcal{N}_t \left(\{r_s, w_s\}_{s=0}^{\infty}\right) \tag{A.57}$$

and the planning problem is still to maximize objective (20), but now subject to implementability (A.57). The RSS optimality condition in that case is given by

$$(1 - \delta (1 + r)) \overline{\ell} \left(\text{mrs}\, \epsilon^{\overline{A},r} + \epsilon^{\overline{A},\tau}\right) - \frac{1 - w}{w} \left(-\epsilon^{N,\tau} - \text{mrs}\, \epsilon^{N,r}\right) - \left(\overline{\ell}\, \text{mrs} - 1\right) = 0$$

Substituting in (A.54), (A.55), and (A.56), we find

$$(1 - \delta (1 + r)) \ell \left(\left(1 + \frac{\mu}{w}\right) \text{mrs}\, \epsilon^{A,r} + \epsilon^{A,\tau}\right) - \frac{1 - w}{w} \left(-\epsilon^{N,\tau} - \text{mrs}\, \epsilon^{N,r}\right) - \left(\left(1 + \frac{\mu}{w}\right) \ell\, \text{mrs} - 1\right) = 0$$

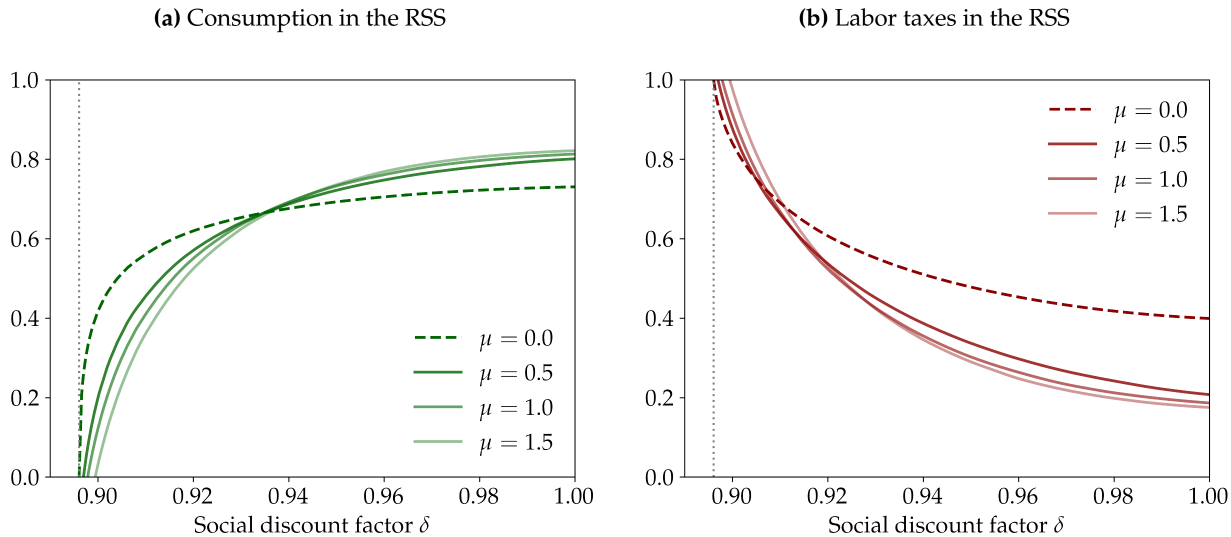
The associated steady state government budget constraint is given by

$$G + r (w + \mu) \mathcal{A}^{ss}(r,1) = (1 - w) \mathcal{N}^{ss}(r,1)$$

Figure F.2 solves for Ramsey steady states as we vary the social discount factor δ between β and 1. For each RSS, panel a shows aggregate consumption and panel b shows labor taxes. The different lines are different sizes μ of the rest of the world. The figure shows that even in an open economy setting, immiseration remains optimal. The intuition for this finding is that even though foreigners cannot be taxed, the government still finds it optimal to sell low-interest-rate debt to foreigners, thereby enabling domestic households to temporarily increase their consumption, even if this comes at the expense of immiseration in the (far) future.

These results are not specific to our Aiyagari economy. They also hold in the BU economy introduced in appendix G.1. More broadly, these results are related to the literature on the optimal provision of safe assets by a hegemon country (see Farhi and Maggiori 2018 and the references therein).

Figure F.2: Ramsey steady states in an open economy



Note: The figures display the interior Ramsey steady state values of labor taxes and aggregate consumption attained as the social discount factor δ is varied from $\delta = \beta$ (grey-dotted vertical line) to $\delta = 1$ for different rest-of-world population masses μ.

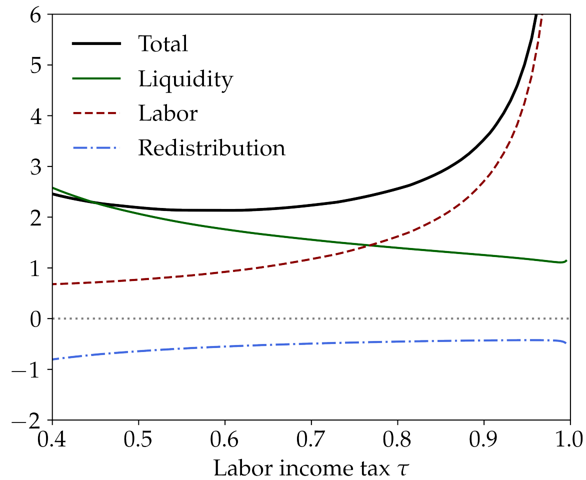
F.4 Private borrowing

For our final subsection, we relax the assumption of a borrowing constraint of zero in the household problem (3). Instead, we assume that the borrowing constraint is given by

$$\underline{a}_t = -\frac{\underline{e}\, w_t\, \bar{n}}{\beta^{-1} - 1}$$

We choose this form as it naturally scales with the average post-tax wage w. $\underline{e}$ is the lowest productivity in the Markov chain, $\overline{n}$ is chosen to be equal to 1, consistent with the optimal labor supply of the borrowing-constrained, lowest-productivity household under the baseline calibration. Figure F.3 shows that even with a relaxed borrowing constraint, the net benefit from greater liquidity is still positive above a labor tax of around 25%. It is not surprising that little is change as we could already see in figure 5 that household liquidity ℓ becomes very large as $\delta \searrow \beta$, in which case most households will be far away from their borrowing constraints anyway.

Figure F.3: Net benefit from higher liquidity with relaxed borrowing constraint



Note: This figure displays the three terms of the interior Ramsey steady state optimality condition (stated in Proposition 3) and their sum for a modified calibration where households have a non-zero borrowing constraint, given by $\underline{a}_t = -\frac{\underline{e}\, w_t \bar{n}}{\beta^{-1}-1}$ where $\underline{e} \approx 0.07$ and $\overline{n} = 1$.

G Details on alternative household sides

G.1 Bonds in utility

In the bonds in utility (BU) model, households maximize

$$\sum_{t=0}^{\infty} \beta^t u(c_t, n_t, a_t)$$

subject to the budget constraint

$$c_t + a_t = (1 + r_t)\, a_{t-1} + w_t n_t \tag{A.58}$$

We work with the utility function (51)

$$u(c, n, a) = \log c_t - \phi \frac{n^{1+\nu}}{1+\nu} + \chi \log a_t \tag{A.59}$$

Optimal household behavior is then determined by the following system of equations: The Euler equation

$$\frac{1}{c_t} = \beta\,(1 + r_{t+1})\, \frac{1}{c_{t+1}} + \chi \frac{1}{a_t} \tag{A.60}$$

the first-order condition for labor supply

$$\phi n_t^{\nu} = \frac{1}{c_t} w_t \tag{A.61}$$

and the budget constraint (A.58). In a steady state with constant r, w, we have

$$(1-\beta(1+r))\frac{a}{c} = \chi$$

$$c = ra + wn$$

$$\phi n^{\nu} c = w$$

Next, we log-linearize the three optimality conditions. First, the Euler equation (A.60)

$$-d\log c_t = \beta(1+r)\left(d\log(1+r_{t+1}) - d\log c_{t+1}\right) + (1-\beta(1+r))\left(-d\log a_t\right) \tag{A.62}$$

Then, the first order condition for labor supply (A.61)

$$\nu d\log n_t = -d\log c_t + d\log w_t$$

and the budget constraint

$$cd\log c_t + ad\log a_t \leq (1+r)\,a\left[d\log(1+r_t) + d\log a_{t-1}\right] + nw\left(d\log w_t + d\log n_t\right)$$

We also express flow utility as

$$du_t = d\log c_t - \phi n_t^{\nu} dn_t + d\log a_t$$

How to derive relationships between discounted elasticities from first order conditions. We can use the following three steps to derive relationships between discount elasticities around the steady state. We explain the steps with the example of the Euler equation (A.62). As a first step, we differentiate both sides w.r.t. an interest rate change dr_s, for some date s,

$$-\frac{d\log c_t}{dr_s} = \beta(1+r)\left(1_{\{s=t+1\}}\frac{1}{1+r} - \frac{d\log c_{t+1}}{dr_s}\right) + (1-\beta(1+r))\left(-\frac{d\log a_t}{dr_s}\right)$$

As a second step, we multiply this condition by δ^{t-s} on both sides,

$$-\delta^{t-s}\frac{d\log c_t}{dr_s} = \beta(1+r)\left(\delta^{t-s}1_{\{s=t+1\}}\frac{1}{1+r} - \delta^{t-s}\frac{d\log c_{t+1}}{dr_s}\right) + (1-\beta(1+r))\left(-\delta^{t-s}\frac{d\log a_t}{dr_s}\right)$$

As a third step, we sum all terms across t,

$$-\sum_{t=0}^{\infty}\delta^{t-s}\frac{d\log c_t}{dr_s} = \beta(1+r)\left(\sum_{t=0}^{\infty}\delta^{t-s}1_{\{s=t+1\}}\frac{1}{1+r} - \sum_{t=0}^{\infty}\delta^{t-s}\frac{d\log c_{t+1}}{dr_s}\right) + (1-\beta(1+r))\left(-\sum_{t=0}^{\infty}\delta^{t-s}\frac{d\log a_t}{dr_s}\right)$$

Changing summation indices to $s+h$, where h runs from $-s$ to ∞ and taking the limit $s \to \infty$, we find

$$-\underbrace{\sum_{h=-s}^{\infty} \delta^h \frac{d \log c_{s+h}}{dr_s}}_{\to \epsilon^{c,r}} = \beta(1+r) \underbrace{\left(\sum_{h=-s}^{\infty} \delta^h 1_{\{h=-1\}} \frac{1}{1+r} - \sum_{h=-s}^{\infty} \delta^h \frac{d \log c_{s+h+1}}{dr_s} \right)}_{\to \frac{1}{\delta(1+r)} - \delta^{-1} \epsilon^{c,r}} + (1-\beta(1+r)) \underbrace{\left(-\sum_{h=-s}^{\infty} \delta^h \frac{d \log a_{s+h}}{dr_s} \right)}_{\to -\epsilon^{a,r}}$$

All in all we find that the Euler equation (A.60) gives rise to the following relationship in terms of discounted elasticities with respect to interest rate changes,

$$\epsilon^{c,r} = \beta(1+r)\left(\delta^{-1}\epsilon^{c,r} - \frac{1}{\delta(1+r)}\right) + (1-\beta(1+r))\,\epsilon^{a,r}$$

Relationships for discounted elasticities. We follow the same steps with all first order conditions and elasticities with respect to both r and w. For the Euler equation, we thus find

$$-\epsilon^{c,r} = \beta(1+r)\left((1+r)^{-1}\delta^{-1} - \delta^{-1}\epsilon^{c,r}\right) - (1-\beta(1+r))\,\epsilon^{a,r}$$

$$-\epsilon^{c,w} = \beta(1+r)\left(-\delta^{-1}\epsilon^{c,w}\right) - (1-\beta(1+r))\,\epsilon^{a,w}$$

For the labor supply condition we find

$$\nu\epsilon^{n,r} = -\epsilon^{c,r}$$

$$\nu\epsilon^{n,w} = -\epsilon^{c,w} + 1$$

For the budget constraint we find

$$c\epsilon^{c,r} + a\epsilon^{a,r} = (1+r)\,a\left(\frac{1}{1+r} + \delta\epsilon^{a,r}\right) + nw\epsilon^{n,r}$$

$$c\epsilon^{c,w} + a\epsilon^{a,w} = (1+r)\,a\delta\epsilon^{a,w} + nw\left(1+\epsilon^{n,w}\right)$$

We can express discounted derivatives of utility as

$$\epsilon^{u,r} = \epsilon^{c,r} - \frac{wn}{c}\epsilon^{n,r} + (1-\beta(1+r))\,\frac{a}{c}\epsilon^{a,r}$$

$$\epsilon^{u,w} = \epsilon^{c,w} - \frac{wn}{c}\epsilon^{n,w} + (1-\beta(1+r))\,\frac{a}{c}\epsilon^{a,w}$$

This leaves us with eight equations and eight unknowns.

Auxiliary objects. Before we derive expressions for the discounted elasticities, we introduce the following auxiliary objects:

$$\hat{\gamma} = \frac{\Delta_\beta}{\Delta_\delta} \frac{r}{((1+r)\beta\delta^{-1} - 1)}$$

$$\gamma^* \equiv \frac{r\beta\delta^{-1}}{(1+r)\beta\delta^{-1} - 1}$$

$$\Delta_\delta \equiv 1 - (1+r)\delta$$

$$\Delta_\beta \equiv 1 - (1+r)\beta$$

Explicit expressions for the discounted elasticities. We find the following expressions:

$$\epsilon^{n,r} = \ell \frac{\hat{\gamma} - \gamma^*}{r\ell\nu(1-\hat{\gamma}) - \hat{\gamma}(1+\nu)}$$

$$\epsilon^{a,r} = \frac{1}{\Delta_\delta} \frac{r\ell\nu(1-\gamma^*) - \gamma^*(1+\nu)}{r\ell\nu(1-\hat{\gamma}) - \hat{\gamma}(1+\nu)}$$

$$\epsilon^{n,w} = \frac{r\ell(1-\hat{\gamma})}{r\ell\nu(1-\hat{\gamma}) - \hat{\gamma}(1+\nu)}$$

$$\epsilon^{a,w} = \frac{r}{\Delta_\delta} \frac{1+\nu}{r\ell\nu(1-\hat{\gamma}) - \hat{\gamma}(1+\nu)}$$

$$\text{mrs} = \frac{\epsilon^{u,w}}{\epsilon^{u,r}} = \frac{1 + \left(\frac{\Delta_\beta}{\Delta_\delta} - 1\right)\Delta_\delta \ell \epsilon^{a,w}}{\ell + \left(\frac{\Delta_\beta}{\Delta_\delta} - 1\right)\Delta_\delta \ell \epsilon^{a,r}}$$

Special case: Infinite Frisch elasticity. Assume now that $\nu = 0$. Then:

$$\epsilon^{n,r} = -\ell \frac{\hat{\gamma} - \gamma^*}{\hat{\gamma}}$$

$$\epsilon^{a,r} = \frac{1}{\Delta_\delta} \frac{\gamma^*}{\hat{\gamma}}$$

$$\epsilon^{n,w} = -\frac{r\ell(1-\hat{\gamma})}{\hat{\gamma}}$$

$$\epsilon^{a,w} = -\frac{r}{\Delta_\delta} \frac{1}{\hat{\gamma}}$$

and the expression for mrs is unchanged. Substituting this expression into our RSS optimality condition (25), we find

$$\gamma^* + r\ell - \frac{1-w}{w}\left(r\ell(\hat{\gamma} - 1) + r\ell\left(\frac{\Delta_\beta}{\Delta_\delta} - 1\right)(\gamma^* - 1) + (\hat{\gamma} - \gamma^*)\right) + \left(\frac{\Delta_\beta}{\Delta_\delta} - 1\right)(\gamma^* + r\ell) = 0$$

Rearranging and solving for $\frac{\tau}{1-\tau} = \frac{1-w}{w}$, we find

$$\frac{\tau}{1-\tau} = \frac{\frac{\Delta_\beta}{\Delta_\delta}\left(\gamma^* + r\ell\right)}{r\ell\left(\hat{\gamma}-1\right) + r\ell\left(\frac{\Delta_\beta}{\Delta_\delta}-1\right)\left(\gamma^*-1\right) + \left(\hat{\gamma}-\gamma^*\right)}$$

Simplifying these terms, we arrive at

$$\frac{\tau}{1-\tau} = \frac{\Delta_\beta}{\delta-\beta}\beta \times \frac{\chi\left(1-\beta^{-1}\delta\right)+\Delta_\beta}{\chi\left(1-\beta\right)+\Delta_\beta} \tag{A.63}$$

which is exactly equal to (52). This gives us the RSS optimality line in the $r-\tau$ diagram in figure 2.

Government budget constraint. The government budget constraint is, as usual,

$$G + r\mathcal{A}^{ss}(r,1)\left(1-\tau\right) = \tau\mathcal{N}^{ss}(r,1) \tag{A.64}$$

To solve for $\mathcal{A}^{ss}(r,1)$ and $\mathcal{N}^{ss}(r,1)$, we combine the BU steady state equations with $\nu = 0$ to find

$$\mathcal{C}^{ss}(r,w) = c = \frac{w}{\phi}$$

$$\mathcal{A}^{ss}(r,w) = a = \phi^{-1}\chi\frac{w}{1-\beta\left(1+r\right)}$$

$$\mathcal{N}^{ss}(r,w) = n = \phi^{-1}\left(1-\chi\frac{r}{1-\beta\left(1+r\right)}\right)$$

Substituting these expressions into (A.64), we arrive at

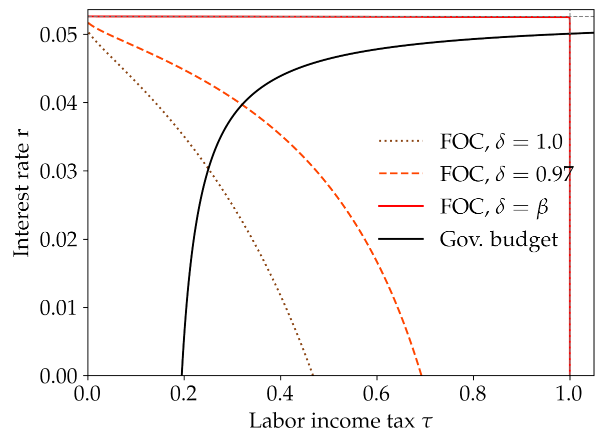
$$\tau = \phi G + \chi\frac{r}{1-\beta\left(1+r\right)} \tag{A.65}$$

The $r-\tau$ diagram for the BU model. Figure G.1 plots the $r-\tau$ diagram for our baseline calibration for various choices of δ. The black solid line is the government budget constraint (A.65); the red dashed line is the RSS optimality condition (A.63). It is straightforward to see that, as we lower δ towards β, the RSS optimality condition steepens and moves to the right, towards higher and higher tax rates. At $\delta = \beta$, the RSS optimality condition is exactly vertical at 100% labor taxes.

Case with positive Frisch elasticity. Next, we show that there is also no RSS in the case where $\delta = \beta$ if the Frisch elasticity ν is positive. With $\delta = \beta$, we have

$$\epsilon^{n,r} = \epsilon^{n,w} = 0 \qquad \epsilon^{a,r} = \frac{1}{1-\left(1+r\right)\beta} \qquad \epsilon^{a,w} = -\frac{r}{1-\left(1+r\right)\beta} \qquad \text{mrs} = \frac{1}{\ell}$$

Figure G.1: Two conditions for the BU economy



Substituting these expressions into our RSS optimality condition (25), we find

$$\underbrace{(1-\beta(1+r))\,\ell\,(\mathrm{mrs}\,\epsilon^{a,r}-\epsilon^{a,w})}_{=1+r\ell>0}-\underbrace{\frac{1-w}{w}\,(\epsilon^{n,w}-\mathrm{mrs}\,\epsilon^{n,r})}_{=0}-\underbrace{(\ell\,\mathrm{mrs}-1)}_{=0}>0$$

Thus, there is no RSS.

G.2 Alternating income states

The Woodford (1990) model assumes that households do not face idiosyncratic income risk, but instead a deterministic sequence of productivities, alternating between $e_{it}=2$ ("employed") and $e_{it}=0$ ("unemployed"). 50% of the population is in each of the two states at any given point in time. Denote by c_{et} consumption of employed households at date t, and by c_{ut} consumption by unemployed households at date t. Denote by a_{et} the saving of employed households at date t. Since unemployed households have no income, they up against a binding borrowing constraint whenever $\beta(1+r)<1$. That is, $a_{ut}=0$ and

$$c_{ut}=(1+r_t)a_{e,t-1}$$

Moreover, only employed households work, $n_{et}>0$ but $n_{ut}=0$. Employed households then solve

$$\max\ \log c_{et}+\beta\log c_{u,t+1}-\phi\frac{n_{et}^{1+\nu}}{1+\nu}$$

subject to

$$c_{et}+\frac{c_{u,t+1}}{1+r_{t+1}}=2w_tn_{et}$$

It is straightforward to see that this implies

$$c_{e,t} = \frac{1}{1+\beta} 2 w_t n_t$$

$$c_{u,t+1} = \frac{\beta}{1+\beta} \left(1 + r_{t+1}\right) 2 w_t n_t$$

$$n_{e,t} = \left(\frac{1+\beta}{\varphi}\right)^{\frac{1}{1+\zeta}}$$

where labor supply is constant, and we denote it by $\overline{n} \equiv n_{e,t}$. Output is therefore constant as well, at $\overline{Y} = \overline{n}$. We therefore have the following aggregate household functions:

$$C_t = \frac{\overline{n}}{1+\beta} w_t + \frac{\beta \overline{n}}{1+\beta} \left(1 + r_t\right) w_{t-1}$$

$$N_t = \overline{n}$$

$$A_t = \frac{\beta \overline{n}}{1+\beta} w_t$$

$$U_t = 0.5 \log w_t + 0.5 \log w_{t-1} + 0.5 \log \left(1 + r_t\right) + const$$

This gives us the following discounted elasticities:

$$\epsilon^{C,w} = \frac{1 + \beta\delta\left(1+r\right)}{1 + \beta\left(1+r\right)} \qquad \epsilon^{C,r} = \frac{\beta}{1+\beta\left(1+r\right)}$$

$$\epsilon^{N,w} = \epsilon^{N,r} = \epsilon^{A,r} = 0 \qquad \epsilon^{A,w} = 1 = -\epsilon^{A,\tau}$$

$$\epsilon^{U,w} = 0.5(1+\delta) \qquad \epsilon^{U,r} = \frac{0.5}{1+r}$$

$$\text{mrs} = \frac{\epsilon^{U,w}}{\epsilon^{U,r}} = \left(1+\delta\right)\left(1+r\right)$$

Moreover, liquidity is given by

$$\ell = \frac{\beta}{1+\beta}$$

Substituting these expressions into the RSS optimality condition (25), we arrive at

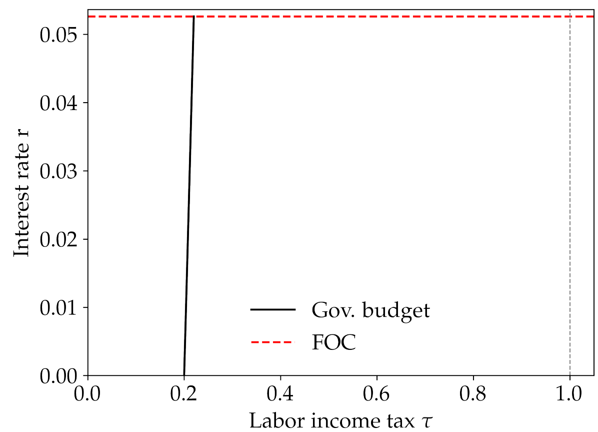
$$-\left(1 - \delta\left(1+r\right)\right) \frac{\beta}{1+\beta} - \frac{\beta}{1+\beta}\left(1+\delta\right)\left(1+r\right) + 1 = 0$$

which we can rearrange to

$$\beta\left(1+r\right) = 1$$

This implies a horizontal RSS optimality line in the $r - \tau$ diagram, irrespective of the social discount factor δ.

Figure G.2: Two conditions for the economy with alternating income states



The government budget constraint is given by

$$G + r\frac{\beta\overline{n}}{1+\beta}(1-\tau) = \tau\overline{n}$$

which we can rearrange to

$$\tau = \frac{\frac{G}{Y}(1+\beta) + \beta r}{1+\beta(1+r)}$$

confirming the results in section 8.3. Figure G.2 illustrates these conditions in the $r-\tau$ diagram. At the RSS here, households are satiated with liquidity and able to smooth consumption perfectly over time. These results are consistent with those in LeGrand and Ragot (2023, prop. 1), for the special case of a constant curvature in the utility over consumption.

Poverty state. Now consider a situation where a share μ of households is in the permanent poverty state (see appendix B.16). In that case, the mrs term is given by

$$\text{mrs} = \frac{\epsilon^{U,w} + \frac{\mu}{1-\mu}}{\epsilon^{U,r}} = \frac{0.5(1+\delta) + \frac{\mu}{1-\mu}}{\frac{0.5}{1+r}} = (1+\delta)(1+r) + 2\frac{\mu}{1-\mu}(1+r)$$

Now, the RSS optimality condition (25) reads

$$-(1-\delta(1+r))\frac{\beta}{1+\beta} - \frac{\beta}{1+\beta}(1+\delta)(1+r) - 2\frac{\mu}{1-\mu}\frac{\beta}{1+\beta}(1+r) + 1 = 0$$

which simplifies to

$$1+r = \beta^{-1}\left(1+2\frac{\mu}{1-\mu}\right)^{-1}$$

In other words, the presence of households in the poverty state, $\mu > 0$, reduces the interest rate at the RSS to a value strictly below $\beta^{-1} - 1$. In particular, this pushes down the red dashed line in figure G.2. The RSS now no longer involves satiation with liquidity; households are unable to perfectly smooth consumption at the RSS.