Determinacy and Existence in the Sequence Space

Adrien Auclert* Matthew Rognlie† Ludwig Straub‡

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Abstract

We develop a *winding number criterion* for determinacy and existence of solutions in the sequence space. We apply this criterion to heterogeneous-agent New-Keynesian (HANK) models. We demonstrate that, in common applications, our criterion is identical to a simple analytical formula.

1 Introduction

The sequence space has become a powerful way to analyze and solve heterogeneous-agent models (e.g. Auclert, Rognlie and Straub 2023, Boppart, Krusell and Mitman 2018, Auclert, Rognlie and Straub 2020, Auclert, Bardóczy, Rognlie and Straub 2021, Wolf 2021). This is because it allows to solve for first-order aggregate shocks, a la Reiter (2009), without the need to carry around large state spaces or rely on dimensionality reduction (Reiter 2010, Bayer and Luetticke 2020). At the core of this approach is the solution $\mathbf{x}$ to a linear system of equations,

$$\mathbf{J} \cdot \mathbf{x} = \mathbf{y} \tag{1}$$

Here, $\mathbf{J} = [J_{t,s}]$ is a *sequence-space Jacobian*, a matrix with columns and rows indexed by the natural numbers; for example, this could be a matrix that maps small changes in date s income into small changes in date t asset demand by households. The vector $\mathbf{x} = \{x_t\}$ is a sequence of unknowns that we would like to solve for; this could, for example, be the response of aggregate income to a shock. The vector $\mathbf{y} = \{y_t\}$ is a sequence of known shocks; this could, for example, capture

*Stanford University, CEPR and NBER. Email: aauclert@stanford.edu.
†Northwestern University and NBER. Email: matthew.rognlie@northwestern.edu.
‡Harvard University, CEPR and NBER. Email: ludwigstraub@fas.harvard.edu.

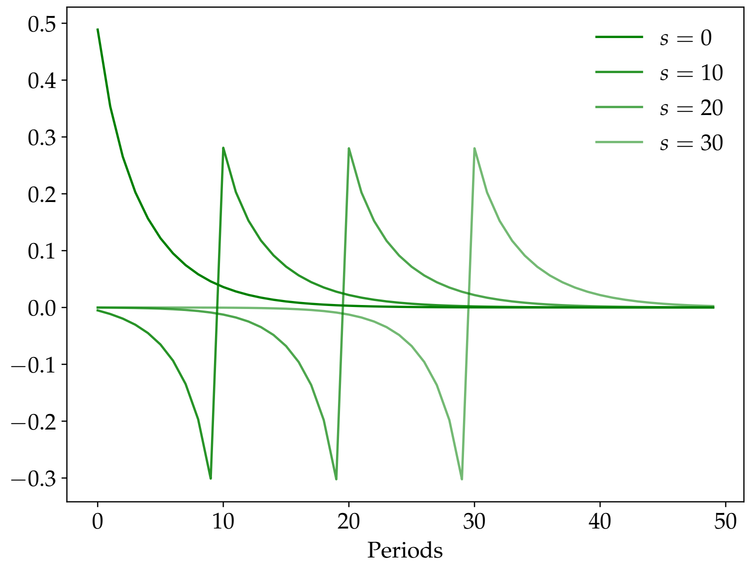


Figure 1: The columns of the Jacobian capturing the asset demand response to income shocks

an increase in asset supply resulting from expansionary fiscal policy. Mathematically, $\mathbf{x}$ and $\mathbf{y}$ are square-summable sequences and $\mathbf{J}$ is a linear operator on square-summable sequences.

In this paper, we ask two questions: First, for which $\mathbf{J}$ does a solution $\mathbf{x}$ to (1) *exist*? And second, for which $\mathbf{J}$ is the solution *unique*, or in other words, *(locally) determinate*?

For general sequence-space Jacobians $\mathbf{J}$, there is no simple condition for existence and uniqueness short of assuming the result, namely that $\mathbf{J}$ be a surjective operator (for existence) and $\mathbf{J}$ be an injective operator (for uniqueness). The starting point of our paper is to restrict attention to a specific class of sequence-space Jacobians $\mathbf{J}$, known as *quasi-Toeplitz operators*. These are Jacobians whose columns converge to a fixed pattern as we go down along the diagonal towards the bottom right. This property captures the idea that the date 90 asset demand response to an income shock at date 100 is nearly identical to the date 190 asset response to a date 200 income shock. We denote by j_k the the date date $s+k$ response to a date s shock for large s, that is, $j_k = \lim_{t\to\infty} J_{s+k,s}$, for any integer k. A very useful way to represent the entries $\{j_k\}$ is as a complex function, a *Laurent series* or *z-transform*, $j(z) \equiv \sum_{k=-\infty}^{\infty} j_k z^k$ for complex z. This function is formally known as the *symbol* of $\mathbf{J}$.

Figure 1 illustrates the quasi-Toeplitz property of the Jacobian of asset demand with respect to income in a standard heterogeneous-agent model (see section 4 for details). The figure plots several columns $J_{\cdot,s}$ of this Jacobian. For each s, column $J_{\cdot,s}$ gives the impulse response of asset demand to the date-0 announcement that income will increase at date s for one period. We see that the impulse response to an unanticipated income shock ($s = 0$) looks quite different from the response to an anticipated shock at date $s = 10$. However, the response to a date $s = 20$ shock looks very similar to the response to a date $s = 30$ shock, except that it is shifted to the right by 10 periods. This is the key feature of a quasi-Toeplitz Jacobian. The entries $\{j_k\}$ in this case

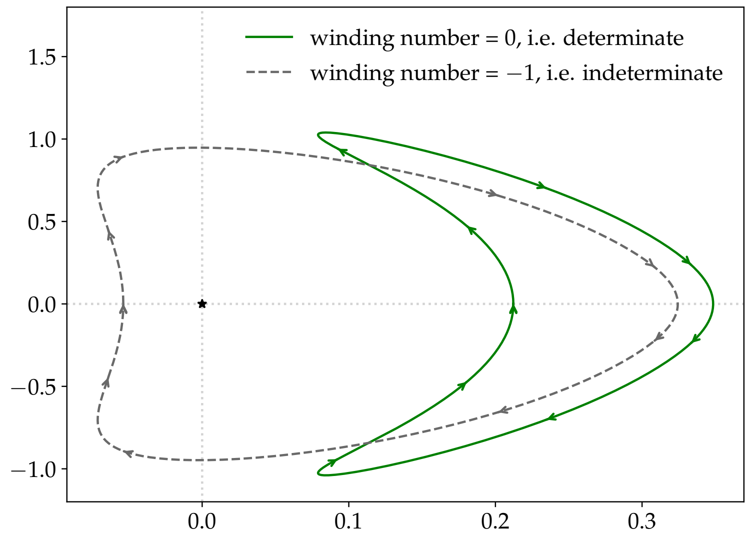


Figure 2: The curve in the complex plane described by $j(z)$ evaluated along the unit circle

correspond to the asymptotic responses of asset demand at date $s+k$ for an income shock at a far-out date s.

The first contribution of this paper is to propose a new criterion for existence and determinacy of a solution $\mathbf{x}$ to (1) based on the symbol $j(z)$. Evaluating $j(z)$ along the complex unit circle, $z = e^{2\pi i\theta}$ for $\theta \in [0,1]$, $j(z)$ describes a simple closed curve in the complex plane. If this curve hits zero for any θ, our sequence-space Jacobian $\mathbf{J}$ is not invertible. If it does not hit zero, we can count how many times it "wraps around" zero in a counter-clockwise fashion, counting clockwise circling as negatives (see figure 2 for an example). This object is known as the *winding number* of $j(z)$. Our criterion states that the sequence-space Jacobian $\mathbf{J}$ is generically injective, ensuring determinacy, if and only if the winding number of $j(z)$ is greater or equal to zero; and that the sequence-space Jacobian $\mathbf{J}$ is generically surjective, ensuring existence, if and only if the winding number of $j(z)$ is less or equal to zero. $\mathbf{J}$ is generically invertible when the winding number of $j(z)$ is zero.

The second contribution of this paper is to show that most sequence-space Jacobians encountered in modern macroeconomic models are indeed quasi-Toeplitz. This is because the Jacobians of individual model elements, or "blocks", are themselves quasi-Toeplitz; and the composition of quasi-Toeplitz Jacobians remains quasi-Toeplitz. While the latter result is well-known in the mathematics literature (e.g. Böttcher and Grudsky 2005a), the former is new to this paper. Specifically, we show that as long as a heterogeneous-agent model is stationary, its Jacobians are naturally quasi-Toeplitz. The repetitive nature of the Jacobian shown in figure 1 is therefore a natural outcome in a wide class of heterogeneous-agent models.

We use our winding number criterion in two applications. First, we consider an analytical model with bonds in the utility function, and show that the winding number criterion coincides with the determinacy and existence criterion that we can obtain through standard methods. Second, we consider a heterogeneous-agent New-Keynesian (HANK) model a la Kaplan, Moll and Violante (2018) and Auclert et al. (2023), and use the criterion to describe the parameter space for which this model admits a determinate solution. We conjecture an analytical Taylor principle for our HANK economy and show this Taylor principle delivers the correct solution in our application. Interestingly, even absent cyclical income risk, the Taylor principle allows for parameter regions in which nominal interest rate pegs deliver determinate solutions.

In future iterations of this note, we plan to study several extensions of our criterion. For instance, our baseline criterion focuses on the case in which the unknown $\mathbf{x} = \{x_0, x_1, \ldots\}$ is single-dimensional within each period, $x_t \in \mathbb{R}$. In an extension, we plan to allow x_t to be an m-dimensional vector with $m > 1$. In that case, $\mathbf{J}$ is naturally quasi *block*-Toeplitz, consisting of small $m \times m$ matrix blocks $\mathbf{J}_{s+k,s}$ which, asymptotically in s, converge to matrices $\mathbf{j}_k$. Now, $\mathbf{J}$'s symbol $\mathbf{j}(z) = \sum_{k=-\infty}^{\infty} \mathbf{j}_k z^k$ is matrix valued. We conjecture that the winding number criterion still works, only that the winding number of the determinant of $\mathbf{j}(z)$ is the correct one to use.

Our result builds on the previous work by Onatski (2006), who presents a largely overlooked way to check determinacy and generic existence using the winding number for standard rational expectations models in the *state space*. To apply the Onatski (2006) criterion, $\mathbf{x} = \{x_0, x_1, \ldots\}$ needs to contain the entire state of the model in every entry x_t. In larger models, this makes x_t m-dimensional, with potentially very large m. Once all states are represented in $\mathbf{x}$, $\mathbf{J}$ is no longer quasi but *exactly* block-Toeplitz, $\mathbf{J}_{s+k,s} = \mathbf{j}_k$ for any s, not just in the limit for large s. In this case, Onatski (2006) shows that the same winding number criterion applies. Our contribution relative to Onatski (2006) is to show that the winding number criterion applies to *quasi*-Toeplitz matrices $\mathbf{J}$ as well. This is important for the analysis of heterogeneous-agent models, where the state space is very large, possibly infinitely dimensional, making it prohibitively costly to stack the entire state in each x_t; and it is important for all models, with or without heterogeneity, that are solved in the sequence space using current methods (Auclert et al. 2021), in which Jacobians are almost never exactly Toeplitz.

Layout. The layout of this paper is as follows. Section 2 starts with equation (1) and derives our main result, the winding number criterion for quasi-Toeplitz matrices $\mathbf{J}$. Section 3 then shows that quasi-Toeplitz matrices indeed emerge very naturally in models that are being analyzed in the sequence space, including stationary heterogeneous-agent models. We discuss the two applications in section 4. We discuss practical implementation considerations in section 5, and conclude in section 6.

2 Generic Existence and Uniqueness Result

The general setting of our paper is that of a general linear system of equations in the sequence space,

$$\mathbf{J} \cdot \mathbf{x} = \mathbf{y} \tag{2}$$

where $\mathbf{y} = \{y_0, y_1, \ldots\} \in \ell^2$ is a given square-summable sequence;[1] $\mathbf{J} = [J_{t,s}] \in \mathbb{R}^{\mathbb{N}\times\mathbb{N}}$ is a quasi-Toeplitz sequence-space Jacobian;[2] and $\mathbf{x} = \{x_0, x_1, \ldots\} \in \ell^2$ is an unknown we wish to characterize. In the following, we formally introduce quasi-Toeplitz Jacobians, their symbols, and prove the winding number criterion for the existence and uniqueness of a solution $\mathbf{x}$ to (2).

2.1 Sequence space

We work in the Hilbert space of real, square-summable sequences ℓ^2 with the usual inner product. We denote the associated norm by $\|\cdot\|$, so for sequence $\mathbf{x} = \{x_0, x_1, \ldots\} \in \ell^2$ we have

$$\|\mathbf{x}\|^2 = \sum_{t=0}^{\infty} x_t^2 < \infty$$

Any sequence $\mathbf{x} \in \ell^2$ necessarily converges to zero. This means that if there exist multiple solutions to (2), there necessarily is local indeterminacy in the sense of Woodford (2003). Vice versa, if there is a unique solution to (2) in ℓ^2, we treat this as indication of local determinacy.[3] Note that uniqueness in ℓ^2, just like local determinacy, does not rule out "explosive solutions" to (2) that lie outside of ℓ^2.

2.2 Quasi-Toeplitz operators

We make two assumptions on the matrix $\mathbf{J}$ motivated by the types of matrices encountered in the analysis of models in the sequence space. First, we assume that $\mathbf{J}$ is a well-defined bounded linear operator on ℓ^2. That is, for each sequence $\mathbf{x} \in \ell^2$, we require that: *(i)* $\sum_{s=0}^{\infty} J_{t,s} \cdot x_s$ is well defined for each $t \in \mathbb{N}$, and square summable across t; and *(ii)* that $\|\mathbf{J} \cdot \mathbf{x}\|$ is bounded above by $M \cdot \|\mathbf{x}\|$ for some $M > 0$ that is independent of $\mathbf{x}$.

Second, we assume that $\mathbf{J}$ is *quasi-Toeplitz.* We define this property in two steps, beginning with Toeplitz operators.

Definition 1. A bounded linear operator $\mathbf{J}$ on ℓ^2 is *Toeplitz* if it can be written as a matrix with

[1] ℓ^2 is the space of all square-summable sequences.

[2] We follow the convention that 0 is included in the natural numbers $\mathbb{N}$.

[3] We are not aware of any examples of sequences $\mathbf{x}$ that converge to zero, are not in ℓ^2, and solve a sequence-space equation like (2) with quasi-Toeplitz $\mathbf{J}$.

constant entries along the diagonals

$$T(\mathbf{j}) \equiv \begin{pmatrix} j_0 & j_{-1} & j_{-2} & \\ j_1 & j_0 & j_{-1} & \\ j_2 & j_1 & j_0 & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix}$$

where the entries $\mathbf{j} = \{j_k\}_{k=-\infty}^{\infty}$ are absolutely summable, $\sum_{k=-\infty}^{\infty} |j_k| < \infty$.

A bounded linear operator $\mathbf{J}$ is Toeplitz if it can be represented as a matrix that is, in some sense, "translation invariant" along the diagonal, $J_{s+k,s} = j_k$. Toeplitz operators (or matrices) are uniquely identified by the two-sided sequence $\mathbf{j} = \{j_k\}_{k=-\infty}^{\infty}$.

As we demonstrate in section 3, sequence-space Jacobians are typically not Toeplitz operators. However, it turns out that they are "close" to being Toeplitz, in the following sense.

Definition 2. A bounded linear operator $\mathbf{J}$ on ℓ^2 is *quasi-Toeplitz* if it can be written as

$$\mathbf{J} = T(\mathbf{j}) + \mathbf{E}$$

where $T(\mathbf{j})$ is the Toeplitz matrix corresponding to some absolutely summable two-sided sequence $\mathbf{j} = \{j_k\}_{k=-\infty}^{\infty}$; and $\mathbf{E}$ is a compact operator on ℓ^2 known as the *(compact) correction.*

Quasi-Toeplitz operators $\mathbf{J}$ are Toeplitz up to a compact operator $\mathbf{E}$ (the "compact correction"). Compact operators are the closure of all finite-dimensional operators on ℓ^2. This means that quasi-Toeplitz operators are Toeplitz except for a correction term that mostly affects $\mathbf{J}$ in finitely many columns and rows and is small otherwise. One formal consequence of this is that $J_{s+k,s}$ is no longer exactly constant and equal to j_k. Instead, for a quasi-Toeplitz Jacobian $\mathbf{J}$ we have that

$$\lim_{s\to\infty} J_{s+k,s} = j_k$$

for all $k \in \mathbb{Z}$.

To illustrate the quasi-Toeplitz nature of actual sequence-space Jacobians, figure 1 plots the columns of the Jacobian of assets with respect to income for the standard heterogenous-agent household side that we'll describe more in section 4 below. Each column s of Jacobian $\mathbf{J}$ corresponds to the impulse response of aggregate asset demand to a one time expected change in income at date s. We see that, as s gets larger, the impulse response converges to a regular pattern *relative* to time s. This pattern is exactly given by $\lim_{s\to\infty} J_{s+k,s} = j_k$. The example illustrates how the compact correction $\mathbf{E}$, which does lead to noticeable deviations from translation invariance for earlier columns (smaller s), dies out for larger s.

2.3 Symbol

The asymptotic behavior of a quasi-Toeplitz operator **J**, described by the two-sided sequence $\mathbf{j} = \{j_k\}_{k=-\infty}^{\infty}$, will be the crucial determinant of whether (2) can be solved or not; and if it can, how many solutions exist. A useful way to represent the two-sided sequence is as complex-valued function over the complex unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. We define the *symbol* of quasi-Toeplitz operator **J** as the complex-valued function $j : \mathbb{T} \to \mathbb{C}$, defined by the Laurent series

$$j(z) \equiv \sum_{k=-\infty}^{\infty} j_k z^k$$

The series is convergent since **j** is absolutely summable by definition 2. One way to think about $j(z)$ is that it is the z-transform of the two-sided sequence **j**.

A very useful property of quasi-Toeplitz matrices is that the product of two such matrices, say **J** and $\tilde{\mathbf{J}}$, with symbols $j(z)$ and $\tilde{j}(z)$, respectively, is itself quasi-Toeplitz, with symbol $j(z) \cdot \tilde{j}(z)$ (see section 3.1 for more details). If we treat the identity matrix as a Toeplitz matrix, its symbol would simply be a constant, $j(z) = 1$.

Applied to our context, this means that for any quasi-Toeplitz sequence-space Jacobian **J** with symbol $j(z)$, its inverse, which will also be quasi-Toeplitz, must have symbol $j(z)^{-1}$. This insight is already quite powerful: We can derive the asymptotic structure of the inverse of **J**—provided **J** is invertible—without doing any complicated math, simply by computing the reciprocal of the symbol. For $j(z)^{-1}$ to be well-defined, it must be that $j(z) \neq 0$—a first necessary condition for invertibility of **J**.

2.4 Winding number

One way to represent a symbol $j(z)$ is to draw its image $j(\mathbb{T})$ in the complex plane. Given that $\mathbb{T}$ is the unit circle and j is continuous, $j(\mathbb{T})$ is a closed curve in $\mathbb{C}$. If $j(z)$ never attains zero, we can define the following object.

Definition 3. The *winding number* $\text{wind}(j)$ of a continuous function $j : \mathbb{T} \to \mathbb{C} \setminus \{0\}$ defined on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is the number of times the graph of $j(z)$ rotates counterclockwise around zero as z goes counterclockwise around $\mathbb{T}$. Mathematically, for each $z = e^{i\theta} \in \mathbb{T}$, $\theta \in [0, 2\pi)$, we can write $j(z)$ as the product of the absolute value $|j(e^{i\theta})|$ and $e^{i\alpha(\theta)}$, with some continuous function $\alpha : [0, 2\pi) \to \mathbb{R}$ characterizing the angle of the complex number $j(e^{i\theta})$. The winding number is given by the integer

$$\text{wind}(j) = \frac{1}{2\pi} \left[\lim_{\epsilon \to 0} \alpha(2\pi - \epsilon) - \alpha(0) \right]$$

In words, the winding number of a closed complex curve $j(z)$ counts how many times the curve loops around zero, with counterclockwise loops counting $+1$ and clockwise loops counting -1.

To see how this works, consider the *lag* and *forward* matrices

$$\mathbf{L} = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix} \qquad \mathbf{F} = \begin{pmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix} \tag{3}$$

Both matrices are clearly quasi-Toeplitz; in fact, they are exactly Toeplitz with a zero correction matrix. The symbol of the lag matrix is $j(z) = z$. Its curve in the complex plane simply moves along a counterclockwise circle, so its winding number is 1. The symbol of the forward matrix is $j(z) = z^{-1}$, describing a clockwise circle in the complex plane, with winding number -1. The identity matrix has a winding number of 0.

Imagine the Jacobian in the linear system (2) is the lag matrix $\mathbf{L}$. That is, the equation we would like to solve reads

$$\mathbf{L} \cdot \mathbf{x} = \mathbf{y} \tag{4}$$

or, written without vector notation, $x_t = y_{t+1}$ for $t \geq 0$. Clearly, this pins down a *unique* candidate solution for $\mathbf{x}$. Mathematically, this means that $\mathbf{L}$ is clearly injective. Is $\mathbf{x}$ always a solution to (4), however? Unfortunately not, as any $\mathbf{y}$ with a nonzero first entry $y_0 \neq 0$ can never be in the range of $\mathbf{L}$. Mathematically, $\mathbf{L}$ is not surjective. Hence, $\mathbf{L}$ is not invertible.

Now imagine the Jacobian in the linear system (2) is the forward matrix $\mathbf{F}$. In that case, $x_{t+1} = y_t$ for $t \geq 0$. This fails to pin down a unique candidate solution, as x_0 is not determined. However, for any $\mathbf{y}$, we can find a solution $\mathbf{x}$. $\mathbf{F}$ is not injective, but surjective, and therefore also not invertible.[4]

We next show that the injectivity and surjectivity properties of the lag matrix $\mathbf{L}$, the forward matrix $\mathbf{F}$, and the identity matrix $\mathbf{I}$ are directly related to the winding numbers of $\mathbf{L}$, $\mathbf{F}$, and $\mathbf{I}$. In fact, *any* quasi-Toeplitz matrix with a positive winding number will (generically) be injective but never surjective, just like $\mathbf{L}$; *any* quasi-Toeplitz matrix with a negative winding number will (generically) be surjective but never injective, just like $\mathbf{F}$; *any* quasi-Toeplitz matrix with a zero winding number will (generically) be both surjective and injective, and thus generically invertible, like $\mathbf{I}$.

2.5 Main result: Winding number criterion

Before we state our main result, we need to describe what we mean by the word "generic": We say that a property holds for *generic* correction matrices $\mathbf{E}$ if it holds on an open and dense subset of compact $\mathbf{E}$'s in the operator norm. This implies that a) whenever that the property does not hold, there exists an arbitrarily small perturbation of $\mathbf{E}$ such that it does hold, and b) whenever it does hold, then it also holds for all other operators in some neighborhood of $\mathbf{E}$. Similarly, we say that

[4]Note that $\mathbf{F}$ is not the inverse of $\mathbf{L}$ (or vice versa). $\mathbf{F}$ is only a "pseudo-inverse" because while $\mathbf{FL} = \mathbf{I}$, the converse is not true, $\mathbf{LF} \neq \mathbf{I}$.

a property holds for a *generic* quasi-Toeplitz matrix $\mathbf{J}$ if it holds for $\mathbf{J} + \mathbf{E}$ with generic correction matrix $\mathbf{E}$. This is a common notion of genericity in mathematics and economics (see, e.g., Onatski 2006).

Our main result is then the following proposition, which provides us with a general relationship between the invertibility properties of quasi-Toeplitz matrices and the winding numbers of their symbols.

Proposition 1. *Consider a quasi-Toeplitz matrix $\mathbf{J}$ with symbol $j(z) = \sum_{k=-\infty}^{\infty} j_k z^k$, that is nonzero along the unit circle, $j(z) \neq 0$ for $z \in \mathbb{T}$. Then:*

a) If $\text{wind}(j) > 0$, $\mathbf{J}$ is not surjective but generically injective: A solution $\mathbf{x} \in \ell^2$ to (2) does not exist for some $\mathbf{y} \in \ell^2$; if it exists, however, it is unique for generic $\mathbf{J}$.

b) If $\text{wind}(j) < 0$, $\mathbf{J}$ is not injective but generically surjective: A solution $\mathbf{x} \in \ell^2$ to (2) exists for any $\mathbf{y} \in \ell^2$ and generic $\mathbf{J}$, but is never unique.

c) If $\text{wind}(j) = 0$, $\mathbf{J}$ is generically invertible[5]: A unique solution $\mathbf{x} \in \ell^2$ to (2) exists for any $\mathbf{y} \in \ell^2$ for generic $\mathbf{J}$.

Proposition 1 directly generalizes the ideas we developed in section 2.4. The winding number exactly determines whether the system (2) has a solution; and if it does, how many solutions there are.

As we will show below, in common applications, it turns out that the threshold between a winding number of zero and a winding number of -1 is determined by the symbol $j(z)$ evaluated at $z = 1$. In those cases, checking the sign of the simple sum of the far-out column, $j(1) = \sum_{k=-\infty}^{\infty} j_k$, is already indicative of determinacy and invertibility.

We prove proposition 1 in the following subsection.[6]

2.6 Proof of proposition 1

We establish three helpful lemmas in appendix A. We build on them to prove each of the three parts of proposition 1 in turn.

Proof of part (c). We begin with the proof of part (c) of proposition 1. Suppose $\mathbf{J}$ is a quasi-Toeplitz matrix with symbol $j(z)$ with zero winding number. Decompose $\mathbf{J}$ as $\mathbf{J} = \mathbf{T}(\mathbf{j}) + \mathbf{E}$. We know from, e.g., Böttcher and Grudsky (2000, Theorem 1.12) that $\mathbf{T}(\mathbf{j})$ is exactly invertible if $j(z)$ has a zero winding number. Thus,

$$\mathbf{J} = \mathbf{T}(\mathbf{j}) \left(\mathbf{I} + \mathbf{T}(\mathbf{j})^{-1} \mathbf{E} \right)$$

[5] In line with the terminology in mathematics, a bounded operator is invertible if it is bijective and its inverse is bounded, too.

[6] Readers not interested in the details of the proof should feel free to skip this subsection.

is generically invertible exactly when $\mathbf{I} + \mathbf{T}(\mathbf{j})^{-1}\mathbf{E}$ is generically invertible. But by Rudin (1991, Theorem 4.18(f)), $\mathbf{T}(\mathbf{j})^{-1}\mathbf{E}$ is a compact operator. Denote by $\tilde{\mathcal{C}}$ the set of compact operators $\mathbf{E}$ for which $\mathbf{I} + \mathbf{E}$ is invertible. Then, the set of compact operators $\mathbf{E}$ for which $\mathbf{I} + \mathbf{T}(\mathbf{j})^{-1}\mathbf{E}$ is invertible must be given by $\mathbf{T}(\mathbf{j}) \cdot \tilde{\mathcal{C}}$, which, using lemma 2, is a dense and open subset of the space of all compact operators $\mathcal{C}$. Thus, $\mathbf{J} = \mathbf{T}(\mathbf{j}) + \mathbf{E}$ is invertible for generic $\mathbf{E}$. This concludes our proof of part (c).

Proof of part (a). Next, consider part (a). We first show that if the winding number of $j(z)$ is positive, $\mathbf{J}$ cannot be surjective. To see this, note that any quasi-Toeplitz operator $\mathbf{J}$ with a symbol $j(z)$ that is nonzero along the unit circle is also a Fredholm operator (Böttcher and Grudsky 2000, Theorem 1.9). The Fredholm index is equal to the negative winding number of the symbol (Böttcher and Grudsky 2000, Theorem 1.9)

$$\text{ind}(\mathbf{J}) = -\text{wind}(j) \tag{5}$$

The Fredholm index is exactly equal to the dimensionality of the kernel of $\mathbf{J}$ minus the dimensionality of its cokernel (Böttcher and Grudsky 2000, Section 1.4)

$$\text{ind}(\mathbf{J}) = \dim \ker\mathbf{J} - \dim \text{coker}\mathbf{J} \tag{6}$$

If the winding number is positive, wind $(j) > 0$, we have that

$$\dim \text{coker}\mathbf{J} > \dim \ker\mathbf{J}$$

implying that the cokernel of $\mathbf{J}$ must have at least dimensionality 1. $\mathbf{J}$ cannot be surjective.

To show that $\mathbf{J}$ is generically injective, consider the operator $\mathbf{J} \cdot \mathbf{F}^{\text{wind}(j)}$ where $\mathbf{F}$ is the forward matrix defined in (3). Recall that $\mathbf{F}$ has a winding number of -1, and that the winding number of a product of quasi-Toeplitz matrices is simply the sum of the winding numbers (see also section 3.1 for more on this point). $\mathbf{J} \cdot \mathbf{F}^{\text{wind}(j)}$ has a winding number of zero and is therefore generically invertible; that is, the set of $\mathbf{E}$ for which $\mathbf{J} \cdot \mathbf{F}^{\text{wind}(j)} + \mathbf{E}$ is invertible is open and dense in $\mathcal{C}$. With $\mathbf{L}$ denoting the lag matrix (3) and using the fact that $\mathbf{F} \cdot \mathbf{L} = \mathbf{I}$, we have

$$\left(\mathbf{J} \cdot \mathbf{F}^{\text{wind}(j)} + \mathbf{E}\right) \mathbf{L}^{\text{wind}(j)} = \mathbf{J} + \mathbf{E}\mathbf{L}^{\text{wind}(j)}$$

is injective for generic $\mathbf{E}$ in $\mathcal{C}$. By lemma 4, since powers of $\mathbf{L}$ are injective and have a closed range, this is equivalent to

$$\mathbf{J} + \mathbf{E}$$

being injective for generic $\mathbf{E}$ in $\mathcal{C}$.

Proof of part (b). We proceed analogously for part (b). First, note that if the winding number of $j(z)$ is negative, $\mathbf{J}$ cannot be injective. This follows from combining (5) and (6): If $\text{wind}(j) < 0$, then

$$\dim \ker \mathbf{J} > \dim \operatorname{coker} \mathbf{J}$$

and so the kernel of $\mathbf{J}$ is nontrivial. $\mathbf{J}$ cannot be injective.

To show that $\mathbf{J}$ is generically surjective, consider the operator $\mathbf{L}^{|\text{wind}(j)|} \cdot \mathbf{J}$. Recall that $\mathbf{L}$ has a winding number of $+1$, so that $\mathbf{L}^{|\text{wind}(j)|} \cdot \mathbf{J}$ has a winding number of zero and is thus generically invertible. That is, $\mathbf{L}^{|\text{wind}(j)|} \cdot \mathbf{J} + \mathbf{E}$ is invertible for generic compact operators $\mathbf{E}$. Since $\mathbf{F}$ is surjective, so is $\mathbf{F}^{|\text{wind}(j)|}$. This means

$$\mathbf{F}^{|\text{wind}(j)|} \left(\mathbf{L}^{|\text{wind}(j)|} \cdot \mathbf{J} + \mathbf{E} \right) = \mathbf{J} + \mathbf{F}^{|\text{wind}(j)|} \mathbf{E}$$

is surjective for generic $\mathbf{E}$. By lemma 4, $\mathbf{J} + \mathbf{E}$ must be surjective for generic $\mathbf{E}$. This proves proposition 1.

2.7 Intuition and relationship to Blanchard-Kahn

We next show how to relate the winding number criterion to standard state-space invertibility criteria. We do so by focusing on the special case where the symbol $\mathbf{j}$ has finitely many nonzero elements. Thus, assume that all elements of $\mathbf{j}$ other than $\{j_{-r}, \ldots, j_s\}$, with $j_{-r} \neq 0$ and $j_s \neq 0$ are zero. This implies that we can write the t-th row of (2) as

$$\sum_{k=-r}^{s} j_k x_{t-k} = y_t \tag{7}$$

This is a standard difference equation. Its characteristic polynomial is exactly given by $j(z)$,

$$j(z) = \sum_{k=-r}^{s} j_k z^k$$

Given that $j(z)$ is assumed to have no zeros on the unit circle, it can be factorized as

$$j(z) = j_s z^{-r} \prod_{n=1}^{N} (z - \delta_n) \prod_{m=1}^{M} (z - \mu_m) \tag{8}$$

where the N roots δ_n lie inside the unit circle, $|\delta_n| < 1$, and the M roots μ_m lie outside the unit circle, $|\mu_m| > 1$. We count roots as many times as their multiplicity, so that

$$M + N = r + s \tag{9}$$

When does the difference equation (7) possess a unique solution? According to the standard Blanchard-Kahn condition, precisely when the number of backward looking terms r in (7) equals the number of stable roots N; and the number of forward looking terms s in (7) equals the number of unstable roots M. Both conditions, $N = r$ and $M = s$ are equivalent given (9).

The Blanchard-Kahn condition is closely related to the winding number criterion. To reveal the connection, consider the Cauchy integral formula for the winding number,

$$\text{wind}(j) = \frac{1}{2\pi i} \oint_j \frac{dz}{z} = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{j'(z)}{j(z)} dz$$

The Cauchy integral formula shows that we can evaluate the winding number simply with the complex path integral, evaluated along the curve $j(z)$. This is useful in our context, because the path integral can be rewritten using the argument principle as[7]

$$\text{wind}(j) = \#\text{ of zeros of } j \text{ inside } \mathbb{T} - \#\text{ of poles of } j \text{ inside } \mathbb{T} \tag{10}$$

With $j(z)$ of the form (8), this exactly shows that

$$\text{wind}(j) = N - r$$

The condition for invertibility in proposition 1, $\text{wind}(j) = N - r = 0$, is thus exactly the same as the condition implied by Blanchard-Kahn. We summarize this in the following corollary.

Corollary 1. *If the number of zeros the symbol $j(z)$ has inside the unit circle $\mathbb{T}$ ...*

a) ... strictly exceeds the number of poles inside $\mathbb{T}$, $\mathbf{J}$ is generically injective but not surjective.

b) ... is strictly less than the number of poles inside $\mathbb{T}$, $\mathbf{J}$ is generically surjective but not injective.

c) ... is equal to the number of poles inside $\mathbb{T}$, $\mathbf{J}$ is generically invertible.

The main difference between our winding number criterion and Blanchard-Kahn is that our criterion is robust to infinitely many non-zero elements in the symbol $\mathbf{j}$. Blanchard-Kahn can no longer be used in this case.

3 Emergence of Quasi-Toeplitz Sequence-Space Jacobians

So far, we have taken the equation (2) with a quasi-Toeplitz matrix $\mathbf{J}$ as given. Now we ask: When is $\mathbf{J}$, in fact, quasi-Toeplitz?

When analyzing macroeconomic models in the sequence space, one often ends up with an equation like (2) as an equilibrium condition. The "total" sequence-space Jacobians $\mathbf{J}$ that enter the equilibrium conditions are often compositions and additions of the "partial" Jacobians of individual model blocks (see Auclert et al. 2021 for details). These blocks are commonly either analytical "simple blocks" or more complex "heterogeneous-agent blocks".

[7] This holds because j in (8) is meromorphic.

In subsection 3.1, we first argue that *if* the "partial" Jacobians of individual model blocks are quasi-Toeplitz, then so are the "total" Jacobians $\mathbf{J}$. Next, we show that partial Jacobians of simple blocks (subsection 3.2) and heterogeneous-agent blocks (subsection 3.3) are indeed quasi-Toeplitz. Together, these results imply that macroeconomic models that consist of simple and heterogeneous-agent blocks necessarily give rise to sequence-space Jacobians $\mathbf{J}$ in equilibrium conditions a la (2) that are quasi-Toeplitz.

3.1 Composition of sequence-space Jacobians

Consider two quasi-Toeplitz matrices $\mathbf{J}, \tilde{\mathbf{J}}$ with symbols $j(z), \tilde{j}(z)$ and corrections $\mathbf{E}, \tilde{\mathbf{E}}$. Then, $\mathbf{J} + \tilde{\mathbf{J}}$ is quasi-Toeplitz with symbol $j(z) + \tilde{j}(z)$ and correction $\mathbf{E} + \tilde{\mathbf{E}}$. It turns out that a similar result holds for multiplication (Böttcher and Grudsky 2005b, Proposition 1.3): $\mathbf{J} \cdot \tilde{\mathbf{J}}$ is quasi-Toeplitz with symbol $j(z) \cdot \tilde{j}(z)$.

This result is very useful in characterizing the "total" Jacobians $\mathbf{J}$ that typically emerge equilibrium conditions of the form (2). This is because, as Auclert et al. (2021) show, these Jacobians are often composed of sums and products of "partial" Jacobians of individual model blocks.

To see how this works, imagine the total Jacobian $\mathbf{J}$ that enters (2) is the Jacobian of aggregate asset demand with respect to aggregate output in a model with a Phillips curve and a standard Taylor rule. This total Jacobian then typically depends on five partial Jacobians:

- Aggregate output influences inflation through a Phillips curve block [one partial Jacobian].
- Inflation and output influence real interest rates via a monetary policy block [two partial Jacobians].
- Real interest rates and output influence asset demand via the household block [two partial Jacobians].

Auclert et al. (2021) show how $\mathbf{J}$ can be computed using additions and products of these partial Jacobians. To ensure that $\mathbf{J}$ is quasi-Toeplitz, each partial Jacobian needs to be quasi-Toeplitz, too. This is what we prove next.

3.2 Jacobians of simple blocks

As in Auclert et al. (2021), we define a *simple block* as a mapping between an input $\mathbf{x} = \{x_0, x_1, x_2, \ldots\}$ and an output $\mathbf{y} = \{y_0, y_1, y_2, \ldots\}$ that is given by a time invariant function h,

$$y_t = h\left(x_{t-k}, \ldots, x_{t+l}\right)$$

for some $k, l \in \mathbb{N}$.[8] The ("partial") Jacobian of y with respect to x of this block around some steady state value $x_t = x_{ss}$ is then given by

$$J_{t,s} = \frac{\partial h}{\partial x_{s-t}} \left(x_{-k} = x_{ss}, \dots, x_{+l} = x_{ss} \right)$$

Since the t, s entry of the Jacobian here only depends on $s - t$, $J_{t+k,t}$ only depends on k. $\mathbf{J}$ is exactly Toeplitz.[9]

3.3 Jacobians of stationary heterogeneous-agent blocks

We define a heterogeneous-agent block as in Auclert et al. (2021). Given the path of an input $\mathbf{x} = \{x_0, x_1, x_2, \dots\}$, a heterogeneous-agent block computes the output path $\mathbf{y} = \{y_0, y_1, y_2, \dots\}$ as the combination of three equations,

$$\mathbf{v}_t = v\left(\mathbf{v}_{t+1}, x_t\right) \tag{11}$$

$$\mathbf{D}_{t+1} = \Lambda\left(\mathbf{v}_{t+1}, x_t\right)' \mathbf{D}_t \tag{12}$$

$$y_t = Y(\mathbf{v}_{t+1}, x_t)' \mathbf{D}_t \tag{13}$$

This system of equations encapsulates three conceptually distinct steps in evaluating a heterogeneous-agent block:

- Equation (11) is a *backward iteration*: it captures how $\mathbf{v}_t$— typically the marginal value function or the policy function—is determined by future $\mathbf{v}_{t+1}$ as well as the current input x_t.
- Equation (12) is a *forward iteration*: it captures how the transition matrix Λ between today's distribution $\mathbf{D}_t$ and tomorrow's distribution $\mathbf{D}_{t+1}$ is determined by next period's marginal value (or policy) function $\mathbf{v}_{t+1}$ as well as the current input x_t.
- Equation (13) is the *measurement equation*: it captures how $\mathbf{v}_{t+1}$ and $\mathbf{D}_t$, together with today's input x_t, determine the output y_t.

We assume that the model is discretized, so that $\mathbf{v}_t$ and $\mathbf{D}_t$ are n-dimensional for some finite n. We comment below on what happens in infinite dimensions. For a given value x_{ss}, the *steady state* of the model is the fixed point $(y_{ss}, \mathbf{v}_{ss}, \mathbf{D}_{ss})$ of (11)–(13) that obtains when $x_t = x_{ss}$ at all times. For convenience, we write $\Lambda_{ss} \equiv \Lambda(\mathbf{v}_{ss}, x_{ss})$ and $\mathbf{Y}_{ss} \equiv Y(\mathbf{v}_{ss}, x_{ss})$. We assume that $v(\cdot), \Lambda(\cdot), Y(\cdot)$ are all continuously differentiable in a neighborhood around the steady state. We denote the derivatives of $v(\cdot)$ at the steady state by $\mathbf{v}_\mathbf{v} \in \mathbb{R}^{n \times n}$ and $\mathbf{v}_x \in \mathbb{R}^{n \times 1}$; the derivatives of $\Lambda(\cdot)' \mathbf{D}_{ss}$ at the steady state by $\Lambda D_\mathbf{v} \in \mathbb{R}^{n \times n}$ and $\Lambda D_x \in \mathbb{R}^{n \times 1}$; and the derivatives of $Y(\cdot)$ at the steady state by $\mathbf{Y}_\mathbf{v} \in \mathbb{R}^{n \times n}$ and $\mathbf{Y}_x \in \mathbb{R}^{n \times 1}$.

[8] One can easily allow there to be multiple inputs and multiple outputs. See Auclert et al. (2021).

[9] It is straightforward to show that $\mathbf{J}$ is also a well-defined linear and bounded operator on ℓ^2 in this case as $J_{t+k,t}$ is only nonzero for finitely many k.

Our goal is to characterize the sequence-space Jacobian $\mathbf{J}$ of $\mathbf{y} = \{y_0, y_1, y_2, \ldots\}$ with respect to $\mathbf{x} = \{x_0, x_1, x_2, \ldots\}$ around the steady state $x_t = x_{ss}$. For this to be well-defined, we make the following assumption.

Definition 4. The heterogeneous-agent block consisting of equations (11)–(13) is *stationary* if *(i)* the Jacobian of $\mathbf{v}_t$ with respect to $\mathbf{v}_{t+1}$, denoted by $\mathbf{v}_\mathbf{v}$, has all eigenvalues inside the unit circle; *(ii)* the steady-state transition matrix Λ_{ss} has all eigenvalues in the unit circle.

Definition 4 captures the very intuitive notion that a heterogeneous-agent model ought to be stationary before we can analyze it. Stationarity requires both that the steady state Markov chain for the distribution is stationary, ensuring a unique steady state distribution $\mathbf{D}_{ss}$; and that the backward iteration is (at least locally) stationary, ensuring a locally unique steady state marginal value (or policy) function $\mathbf{v}_{ss}$.[10]

Fake-news matrix. One way to think about the Jacobian $\mathbf{J}$ of $\mathbf{y}$ to $\mathbf{x}$ is that it is a matrix of *news shocks:* $J_{\cdot,s}$ can be regarded as the impulse response of outcome y to a news shock to x that hits the economy s periods in the future, but is learned about at date 0 already. We next introduce an auxiliary matrix $\mathbf{F}$ which contains all the same information as $\mathbf{J}$, but in a somewhat more tractable format. We call $\mathbf{F}$ the "fake news" matrix (see also Auclert et al. 2021).

Definition 5. Assume the heterogeneous-agent model (11)–(13) is stationary. The *fake-news matrix* $\mathbf{F}$ of $\mathbf{y}$ to $\mathbf{x}$ is defined as follows: The first column is given by

$$F_{t,0} = \begin{cases} \mathbf{Y}_x' \mathbf{D}_{ss} & t = 0 \\ \mathbf{Y}_{ss}' \left(\Lambda_{ss}'\right)^{t-1} \Lambda D_x & t > 0 \end{cases}$$

and all other columns $s > 0$ are defined by

$$F_{t,s} = \begin{cases} \mathbf{v}_x' \left(\mathbf{v}_\mathbf{v}'\right)^{s-1} \mathbf{Y}_\mathbf{v}' \mathbf{D}_{ss} & t = 0 \\ \mathbf{Y}_{ss}' \left(\Lambda_{ss}'\right)^{t-1} \Lambda D_\mathbf{v} \left(\mathbf{v}_\mathbf{v}\right)^{s-1} \mathbf{v}_x & t > 0 \end{cases}$$

The fake news matrix has two separate formulas for the first column $s = 0$ and all other columns $s > 0$. The first column corresponds to the response of outcome y to a one-time unanticipated shock to x at date $t = 0$, that is $F_{t,0} = \partial y_t / \partial x_0$.

The later columns $s > 0$ of $\mathbf{F}$ correspond to the impulse responses of outcome y to "fake" news shocks in x. To illustrate, fix a column $s > 0$, $F_{\cdot,s}$. Consider the following experiment. At date $t = 0$, the news shock that x_s increases marginally is announced. This affects the current marginal value function (or policy) $\mathbf{v}_0$ at date 0, by $(\mathbf{v}_\mathbf{v})^{s-1} \mathbf{v}_x$. Through this change, the average outcome y_0 at date 0 is affected by $\mathbf{v}_x' \left(\mathbf{v}_\mathbf{v}'\right)^{s-1} \mathbf{Y}_\mathbf{v}' \mathbf{D}_{ss}$.

[10]One can analogously state the three equations (11)–(13) of the heterogeneous-agent block with an infinite-dimensional state space. In this case, stationarity can be defined in a similar fashion, with eigenvalues of the two operators inside the unit circle with absolute values bounded away from 1. The results below follow analogously.

At date 1, however, the announced shock is nullified, explaining the "fake" in "fake news". Policies at date 1 and thereafter thus remain unchanged relative to the steady state. This does not mean the shock will have no impact beyond date 0, however. Policies *did* change in period 0 by $(\mathbf{v_v})^{s-1}\mathbf{v}_x$, and that has an effect on average outcomes t periods into the future via the law of motion of the distribution, captured by term $\mathbf{Y}'_{ss}(\Lambda'_{ss})^{t-1}$.

A useful property of the fake news matrix of a stationary heterogeneous-agent block is that its entries decay exponentially in both rows and columns.

Lemma 1. *For a stationary heterogeneous-agent block, we have*

$$|F_{t,s}| \leq C\Delta^{s+t} \tag{14}$$

for some $\Delta \in (0,1)$, $C > 0$.

Proof. Let δ be the largest eigenvalue of $\mathbf{v_v}$ and δ' the largest eigenvalue of Λ_{ss}. Define $\Delta = \max\{\delta, \delta'\}$. Notice that this implies that the matrix norms of $\mathbf{v_v}$ and Λ_{ss} are bounded by Δ. For $s, t > 0$ we then have

$$|F_{t,s}| \leq C_1 \|\Lambda_{ss}\|^{t-1} \|\mathbf{v_v}\|^{s-1} \leq C_2 \Delta^{s+t}$$

for constants $C_1, C_2 > 0$. Similar inequalities bound $F_{t,0}$ and $F_{0,s}$,

$$|F_{t,0}| \leq C_3 \Delta^t \qquad |F_{0,s}| \leq C_4 \Delta^s$$

with $C_3, C_4 > 0$. Defining $C = \max\{C_1, C_2, C_3, C_4\}$, we obtain (14). □

The Jacobian we are ultimately after captures the dependence of y on x, that is,

$$J_{t,s} = \frac{\partial y_t}{\partial x_s}$$

Auclert et al. (2021) show that this Jacobian can be computed quite straightforwardly from the fake news matrix:

$$J_{t,s} = \sum_{u=0}^{\min(t,s)} F_{t-u,s-u} \tag{15}$$

We are ready to state our main result in this subsection.

Proposition 2. *The sequence-space Jacobian* $\mathbf{J}$ *of a stationary heterogeneous-agent block is quasi-Toeplitz*

$$\mathbf{J} = \mathbf{T}(\mathbf{j}) + \mathbf{E} \tag{16}$$

with two-sided sequence $\mathbf{j} = \{j_k\}_{k=-\infty}^{\infty}$ *and correction matrix* $\mathbf{E} = (E_{t,s})$

$$j_k \equiv \sum_{v=\max\{0,-k\}}^{\infty} F_{k+v,v} \qquad \textit{and} \qquad E_{t,s} \equiv -\sum_{u=1}^{\infty} F_{t+u,s+u} \tag{17}$$

Proof. We proceed in two steps. First, we show that $\mathbf{J}$ can indeed be decomposed as in (16). Second, we show that the correction $\mathbf{E}$ is indeed compact.

To show (16), consider entry $J_{s+k,s}$ of the Jacobian $\mathbf{J}$. Applying equation (15), we have

$$J_{s+k,s} = \sum_{u=0}^{\min\{s,s+k\}} F_{s+k-u,s-u} = \sum_{v=\max\{0,-k\}}^{s} F_{k+v,v}$$

where we have used the change of variables $v = s - u$. Given bound (14), this series converges as $s \to \infty$, so that we can write $J_{s+k,s}$ as

$$J_{s+k,s} = \sum_{v=\max\{0,-k\}}^{\infty} F_{k+v,v} - \sum_{v=s+1}^{\infty} F_{k+v,v} \tag{18}$$

The first term in (18) is exactly equal to j_k. The second term can be recast as

$$-\sum_{v=s+1}^{\infty} F_{k+v,v} = -\sum_{u=1}^{\infty} F_{s+k+u,s+u} = E_{s+k,s}$$

This proves that $\mathbf{J}$ can be decomposed as in (16), with $\mathbf{j}$ and $\mathbf{E}$ defined as in (17).

To show that $\mathbf{E}$ is compact, notice that we can use (14) to bound the entries of $\mathbf{E}$ by

$$|E_{t,s}| \leq \sum_{u=1}^{\infty} |F_{t+u,s+u}| = \frac{C\Delta^2}{1-\Delta^2} \cdot \Delta^{s+t} \leq \tilde{C} \cdot \Delta^{s+t}$$

for some $\tilde{C} > 0$. Any operator $\mathbf{E}$ whose entries are bounded in this form are compact. To see this, denote by $\mathbf{E}^{(T)}$ the truncated version of $\mathbf{E}$, that is, an operator on ℓ^2 with the same entries $E_{t,s}$ for $t, s \leq T$ but with zeros elsewhere. $\mathbf{E}^{(T)}$ is clearly compact as it is only non-zero on a finite-dimensional subspace of ℓ^2. We can bound the operator norm difference between $\mathbf{E}$ and $\mathbf{E}^{(T)}$ by

$$\|\mathbf{E} - \mathbf{E}^{(T)}\| \leq \tilde{C} \sum_{s \geq T+1} \sum_{t \geq 0} \Delta^{s+t} + \tilde{C} \sum_{t \geq T+1} \sum_{s \geq 0} \Delta^{s+t} \leq \frac{2\Delta}{1-\Delta} \tilde{C} \cdot \Delta^T$$

Thus, $\mathbf{E}^{(T)} \to \mathbf{E}$ in operator norm. Since the set of compact operators is closed, $\mathbf{E}$ must be compact as well. □

3.4 Steady state shifts

Below it will sometimes useful to compute the sum of the two-sided sequence of a quasi-Toeplitz matrix, that is,

$$\sum_{k=-\infty}^{\infty} j_k = j(1)$$

which is simply the symbol evaluated at value $z = 1$. As it turns out, for both simple and heterogeneous-agent blocks, this is exactly the steady state sensitivity of y_{ss} to x_{ss}.

Proposition 3. *For any simple block or stationary heterogenous-agent block, the derivative of the steady state outcome y_{ss} with respect to the steady state input x_{ss} is given by*

$$\frac{\partial y_{ss}}{\partial x_{ss}} = \sum_{k=-\infty}^{\infty} j_k = j(1) \tag{19}$$

Proof. For a simple block,

$$y_{ss} = h\left(x_{-k} = x_{ss}, \ldots, x_{+l} = x_{ss}\right)$$

and so

$$\frac{\partial y_{ss}}{\partial x_{ss}} = \sum_{u=-k}^{l} j_u = \sum_{u=-\infty}^{\infty} j_k = j(1)$$

For a stationary heterogenous-agent block,

$$\begin{aligned} \mathbf{v}_{ss} &= v\left(\mathbf{v}_{ss}, x_{ss}\right) \\ \mathbf{D}_{ss} &= \Lambda\left(\mathbf{v}_{ss}, x_{ss}\right)' \mathbf{D}_{ss} \\ y_{ss} &= Y(\mathbf{v}_{ss}, x_{ss})' \mathbf{D}_{ss} \end{aligned}$$

After a few lines of algebra, we find

$$\begin{aligned} \frac{\partial y_{ss}}{\partial x_{ss}} &= \mathbf{Y}_x' \mathbf{D}_{ss} + \mathbf{Y}_{ss}' \left(\mathbf{I} - \Lambda_{ss}'\right)^{-1} \Lambda D_x \\ &\quad + \mathbf{v}_x' \left(\mathbf{I} - \mathbf{v}_{\mathbf{v}}'\right)^{-1} \mathbf{Y}_{\mathbf{v}}' \mathbf{D}_{ss} + \mathbf{Y}_{ss}' \left(\mathbf{I} - \Lambda_{ss}'\right)^{-1} \Lambda D_{\mathbf{v}} \left(\mathbf{I} - \mathbf{v}_{\mathbf{v}}\right)^{-1} \mathbf{v}_x \\ &= \sum_{t=0}^{\infty} \sum_{s=0}^{\infty} F_{t,s} \end{aligned}$$

We arrive at the same double sum starting from $\sum_{k=-\infty}^{\infty} j_k$ and substituting in (17),

$$\begin{aligned} \sum_{k=-\infty}^{\infty} j_k &= \sum_{k=-\infty}^{\infty} \sum_{v=\max\{0,-k\}}^{\infty} F_{k+v,v} = \sum_{k=-\infty}^{-1} \sum_{v=-k}^{\infty} F_{k+v,v} + \sum_{k=0}^{\infty} \sum_{v=0}^{\infty} F_{k+v,v} \\ &= \sum_{k=1}^{\infty} \sum_{v=k}^{\infty} F_{v-k,v} + \sum_{k=0}^{\infty} \sum_{v=0}^{\infty} F_{k+v,v} = \sum_{v=1}^{\infty} \sum_{u=0}^{v-1} F_{u,v} + \sum_{v=0}^{\infty} \sum_{u=v}^{\infty} F_{u,v} = \sum_{t=0}^{\infty} \sum_{s=0}^{\infty} F_{t,s} \end{aligned}$$

proving that, indeed,

$$j(1) = \sum_{k=-\infty}^{\infty} j_k = \frac{\partial y_{ss}}{\partial x_{ss}}$$

□

4 Application to HANK

We now consider two types of heterogeneous-agent New-Keynesian models as applications of proposition 1. Both models are introduced in detail in Auclert et al. (2023). With a constant real interest rate, that paper shows that the equilibrium output response $d\mathbf{Y}$ to fiscal policy satisfies two equations. The first, in the goods space, is the intertemporal Keynesian cross,

$$d\mathbf{Y} = d\mathbf{G} - \mathbf{M}d\mathbf{T} + \mathbf{M}d\mathbf{Y}$$

The second, in the asset space, is given by

$$\mathbf{A}d\mathbf{Y} = d\mathbf{B} + \mathbf{A}d\mathbf{T} \tag{20}$$

Here $d\mathbf{T} = \{dT_0, dT_1, \ldots\}$ is the path of tax increases; $d\mathbf{B}$ is the path of changes in government debt; $d\mathbf{G}$ is the path of changes in government spending; $\mathbf{M}$ is the sequence space Jacobian of consumption C with respect to after-tax income Z; and $\mathbf{A}$ is the sequence space Jacobian of asset demand A with respect to after-tax income Z among households in the model. In this paper, we focus on the asset-space equation (20).[11]

The two models we consider are the two-agent bond-in-the-utility ("TABU") model; and second the one-account heterogeneous-agent model ("HA-one"). We discuss the solutions to (20) in each of the models in turn and then consider deviations from constant real interest rates to study the Taylor principle in the heterogeneous-agent economy.

4.1 TABU model at constant r

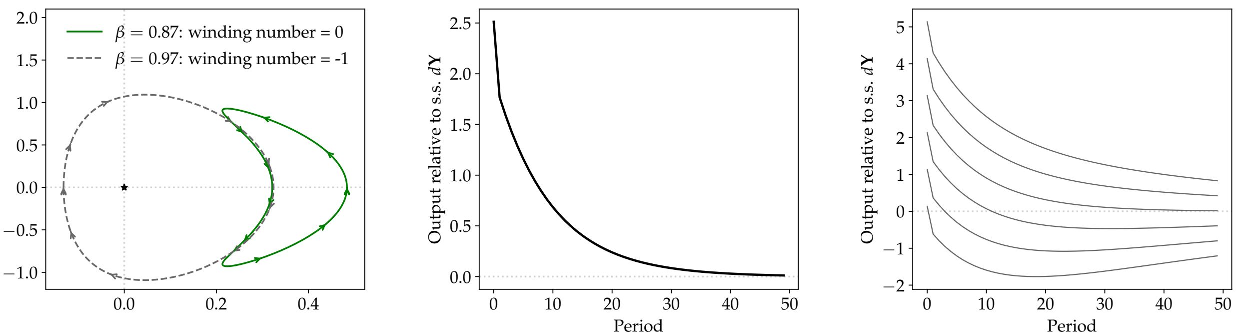
The TABU model consists of two types of households. A share $1-\mu$ of households is forward-looking with discount factor β, unconstrained, and derives utility from consumption and holdings of government bonds ("BU" for bond-in-the-utility). We denote the MPC out of a one-time transfer of that household by $1 - \frac{\lambda}{1+r}$ with $\lambda \in (0, 1+r)$ a parameter that is implicitly pinned down by the preferences of the unconstrained household. The remaining share μ of households is hand-to-mouth.

Auclert et al. (2023) derive the following closed-form expression for the asset jacobian $\mathbf{A}$ of the TABU model,

$$\mathbf{A}^{TABU} = (1-\mu)\begin{pmatrix} 1 & 0 & 0 & \cdots \\ \lambda & 1 & 0 & \cdots \\ \lambda^2 & \lambda & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} \frac{\lambda}{1+r} & -\left(1-\frac{\lambda}{1+r}\right)\cdot\beta\lambda & -\left(1-\frac{\lambda}{1+r}\right)\cdot(\beta\lambda)^2 & \cdots \\ 0 & \frac{\lambda}{1+r} & -\left(1-\frac{\lambda}{1+r}\right)\cdot\beta\lambda & \cdots \\ 0 & 0 & \frac{\lambda}{1+r} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{21}$$

To derive results on existence and determinacy of $d\mathbf{Y}$ in (20) using proposition 1, we need to compute the symbol $a^{TABU}(z)$ of $\mathbf{A}^{TABU}$. As we explained in section 3.1, the symbol of a product of quasi-Toeplitz matrices is the product of the matrices' symbols. Thus, the symbol of $\mathbf{A}^{TABU}$ is

[11] In the language of Auclert et al. (2023), $\mathbf{A} = \mathbf{K}\left(\mathbf{I}-\mathbf{M}\right)$ where $\mathbf{K} = -\sum_{t=1}^{\infty}\left(1+r\right)^{-t}\mathbf{F}$.



(a) Graphs of the symbol (b) Output response (determinate) (c) Output responses (indeterminate)

Figure 3: Determinacy and the response to fiscal policy in the TABU model

Note. Both parameterizations use $\lambda = 0.75$, $\mu = 0.32$, $r = 0.05$. The green solid uses $\beta = 0.87$, the gray dashed uses $\beta = 0.97$.

exactly equal to the product of the symbols of the two matrices in (21) (and $1 - \mu$),

$$a^{TABU}(z) = (1-\mu)\left(\sum_{t=0}^{\infty}\lambda^t z^t\right)\cdot\left(1-\left(1-\frac{\lambda}{1+r}\right)\sum_{t=0}^{\infty}(\beta\lambda)^t z^{-t}\right)$$

which we can further simplify to

$$a^{TABU}(z) = (1-\mu)\frac{1}{1-\lambda z}\cdot\left(1-\left(1-\frac{\lambda}{1+r}\right)\frac{1}{1-\beta\lambda z^{-1}}\right) \tag{22}$$

Panel (a) of figure 3 plots the graph of $a^{TABU}(z)$ in the complex plane for two sets of parameters. Both assume $\lambda = 0.75$, $\mu = 0.32$, $r = 0.05$, as in Auclert et al. (2023). For the discount factor, the first parameterization assumes $\beta = 0.87$ (green solid line) as in Auclert et al. (2023), while the second assumes $\beta = 0.97$ (gray dashed line). We see that the graph does not wrap around zero with the first parameterization, indicating a winding number of zero. Applying proposition 1, this proves (generic) existence and determinacy of the solution to (20). With the second parametrization, the graph does wrap around zero, exactly once in a clockwise fashion, indicating a winding number of -1. According to proposition 1, we therefore generally have existence of a solution, but indeterminacy.

Panel (b) of figure 3 shows the unique output response $d\mathbf{Y} = \{dY_0, dY_1, \ldots\} \in \ell^2$ to a simple fiscal policy shock in the first, determinate parameterization.[12] Panel (c) of figure 3 shows the range of output responses in the second, indeterminate parameterization. As we can see, multiple bounded solutions exist, just as predicted by the winding number criterion.

For the TABU model, we can directly infer the winding number from (22). We do this using formula (10) describing the winding number as the difference between the symbol's number of

[12] The shock is an AR(1) increase in government spending, $dG_t = \rho_G^t dG_0$ where dG_0 is chosen to be 1% of output, and taxes are chosen to ensure an AR(1) path of government debt, $dB_t = \rho_B^t dB_0$. The persistences are $\rho_G = \rho_B = 0.9$.

zeros minus its number of poles inside the unit circle. We can rewrite $a^{TABU}(z)$ as

$$a^{TABU}(z) = (1-\mu)\frac{\lambda}{1+r}\frac{1}{1-\lambda z}\cdot\left(\frac{z-\beta(1+r)}{z-\beta\lambda}\right)$$

This is a rational function with two poles, one at $z = \lambda^{-1} > 1$ and one at $z = \beta\lambda \in (0,1)$. It has one zero, at $z = \beta(1+r)$. It immediately follows that

$$\text{wind}\left(a^{TABU}\right) = \begin{cases} 0 & \text{if } \beta(1+r) < 1 \\ -1 & \text{if } \beta(1+r) > 1 \end{cases}$$

The TABU model is therefore generically determinate under a constant real interest rate rule if $\beta(1+r) < 1$, and indeterminate otherwise.

4.2 Heterogeneous-agent model at constant r

Next we study a one-account heterogeneous-agent (HA) model in which households are subject to uninsurable idiosyncratic income risk. We describe the details of the model in Auclert et al. (2023). We use a version of the model that allows for arbitrary cyclicality of income risk, parameterized by γ. Specifically, we use the formalization in Auclert and Rognlie (2020), and assume that labor supply n_{it} of agent i at date t depends on aggregate labor demand N_t according to

$$n_{it} = N_t \frac{(e_{it})^{\gamma \log N_t}}{\mathbb{E}_i\left[(e_{it})^{\gamma \log N_t}\right]}$$

This formulation recovers the equal-rationing case in Auclert et al. (2023) with acyclical income risk if $\gamma = 0$. If $\gamma > 0$, income risk is procyclical. If $\gamma < 0$, income risk is countercyclical.

Like the TABU model, equation (20) characterizes the general equilibrium output response $d\mathbf{Y}$, where $\mathbf{A} = \mathbf{A}^{HA}$ is now the sequence-space Jacobian of the heterogeneous-agent model's asset demand to after-tax income. $\mathbf{A}^{HA}$ can no longer be computed in closed form and needs to be computed numerically (e.g. using the algorithm in Auclert et al. 2021). Figure 4 shows various columns of $\mathbf{A}^{HA}$ for two values of γ.

To check the existence and determinacy properties of the solution $d\mathbf{Y}$ to (20), we plot the symbol $a^{HA}(z)$ in figure 5, for two different parameterizations of the cyclicality of income risk. The acyclical income risk specification clearly leads to a winding number of zero, indicating generic determinacy and uniqueness. By contrast, the countercyclical income risk specification leads to a winding number of -1, indicating indeterminacy.[13]

Interestingly, as we vary γ from procyclical to countercyclical income risk, the first point on the graph of $a^{HA}(z)$ that crosses zero and causes the winding number to flip from 0 to -1 is the

[13] See Ravn and Sterk (2017), Bilbiie (2019), Acharya and Dogra (2020) for studies of income risk and indeterminacy in tractable heterogeneous-agent models.

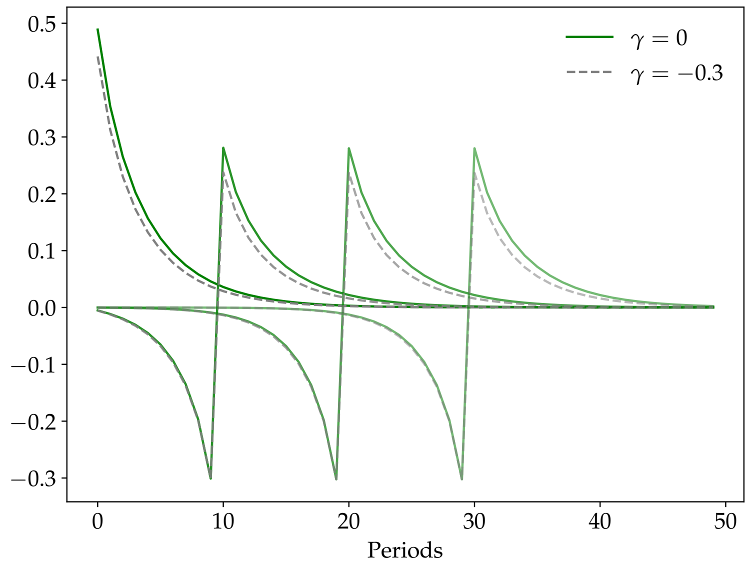


Figure 4: Columns of the het.-agent asset Jacobian, by income risk cyclicality γ

point associated with $z = 1$. Since $a^{HA}(1)$ is exactly the response of steady state asset demand to after-tax income, in practice, for this class of heterogeneous-agent models, we have found that determinacy is ensured exactly when *steady state* assets increase in response to a permanent, *steady state,* upward shift in after-tax income. Very naturally, this is more likely for higher γ (acyclical or even procyclical income risk), as greater after-tax income then goes hand in hand with greater risk, leading to greater asset accumulation. Vice versa, if γ is more negative, greater after-tax income reduces income risk, reducing the incentive to accumulate assets and thus reducing $a^{HA}(1)$, making indeterminacy more likely.

With acyclical income risk, $\gamma = 0$, we can use homotheticity of the household side of the heterogeneous-agent model to express $a^{HA}(1)$ directly as function of steady state objects,

$$a^{HA}(1) = \frac{B_{ss}}{Y_{ss} - T_{ss}} \tag{23}$$

This follows because a one percent increase in after-tax income $Y_{ss} - T_{ss}$ in each state of the world in the steady state translates into a one percent higher steady state asset position B_{ss}. Since $B_{ss} > 0$ in the steady state of the model here, this shows that $a^{HA}(1) > 0$, suggesting determinacy of the heterogeneous-agent model with acyclical income risk under a constant real interest rate rule.

4.3 Taylor rules

One way to achieve determinacy in New-Keynesian (NK) models is via a Taylor rule. To allow for a Taylor rule, we need to specify three additional equations in vector form. The first equation is a

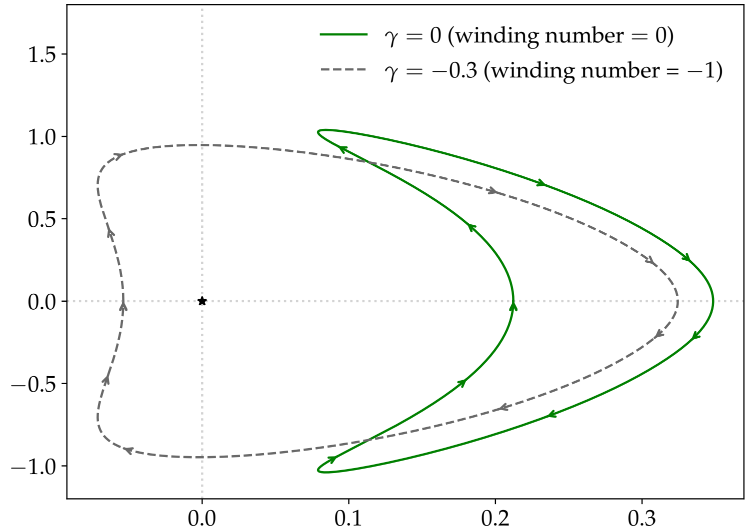


Figure 5: Graphs of the heterogeneous-agent symbol

New-Keynesian Phillips curve (NK-PC). To make our lives simple here, we postulate a relatively simple NK-PC, which relates inflation at date t, π_t, to future output changes dY_{t+s},

$$\pi_t = \kappa \sum_{s=0}^{\infty} \beta^s dY_{t+s}$$

In vector form (see Auclert, Rigato, Rognlie and Straub 2022 for details), this can be written as

$$\boldsymbol{\pi} = \kappa \underbrace{\begin{pmatrix} 1 & \beta & \beta^2 & \ddots \\ 0 & 1 & \beta & \ddots \\ 0 & 0 & 1 & \ddots \\ \ddots & \ddots & \ddots & \ddots \end{pmatrix}}_{\mathbf{K}} d\mathbf{Y} \tag{24}$$

where $\mathbf{K}$ is known as the Generalized Phillips curve matrix.

The second is the sequence-space Jacobian of asset demand to real interest rates, which we denote by $\mathbf{A}^r$. To distinguish it from the Jacobian of asset demand to income, we denote the latter by $\mathbf{A}^Y$. The response of asset demand $d\mathbf{A}$ is then

$$d\mathbf{A} = \mathbf{A}^Y (d\mathbf{Y} - d\mathbf{T}) + \mathbf{A}^r d\mathbf{r} \tag{25}$$

Finally, the third equation combines the Fisher equation with a Taylor rule, requiring that the

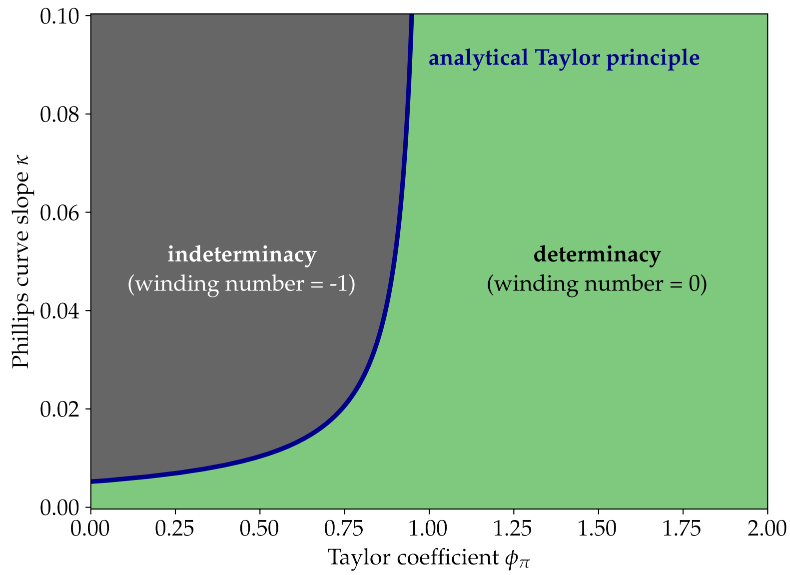


Figure 6: Winding number in the HA model as function of Taylor coefficient ϕ_π and Phillips curve slope κ

nominal interest rate $dr_t + \pi_{t+1}$ moves with current inflation π_t,

$$dr_t + \pi_{t+1} = \phi_\pi \pi_t$$

In vector notation, this equation can be reformulated using the forward matrix $\mathbf{F}$ as

$$d\mathbf{r} = (\phi_\pi \mathbf{I} - \mathbf{F})\, \boldsymbol{\pi} \tag{26}$$

Combining asset market clearing $d\mathbf{A} = d\mathbf{B}$ with (24), (25), and (26) then allows us to generalize (20) to apply to an economy with a Taylor rule,

$$\underbrace{\left(\mathbf{A}^Y + \mathbf{A}^r (\phi_\pi \mathbf{I} - \mathbf{F})\, \mathbf{K}\right)}_{\equiv \mathbf{A}^{Taylor}} d\mathbf{Y} = d\mathbf{B} + \mathbf{A}^Y d\mathbf{T} \tag{27}$$

To check whether a Taylor rule with Taylor coefficient ϕ_π induces determinacy, we simply compute the winding number of $\mathbf{A}^{Taylor} = \mathbf{A}^Y + \mathbf{A}^r (\phi_\pi \mathbf{I} - \mathbf{F})\, \mathbf{K}$.

Figure 6 does this for arbitrary combinations of ϕ_π and κ, in the case of acyclical income risk ($\gamma = 0$). As in the textbook three-equation NK model, greater Taylor rule coefficients make determinacy unambiguously more likely. Different from the textbook NK model, however, the Phillips curve slope parameter κ matters for determinacy even absent an output gap term in the Taylor rule. In fact, for small Phillips curve slope parameters κ, determinacy emerges even if the Taylor coefficient is zero, $\phi_\pi = 0$, indicating a nominal interest rate peg.

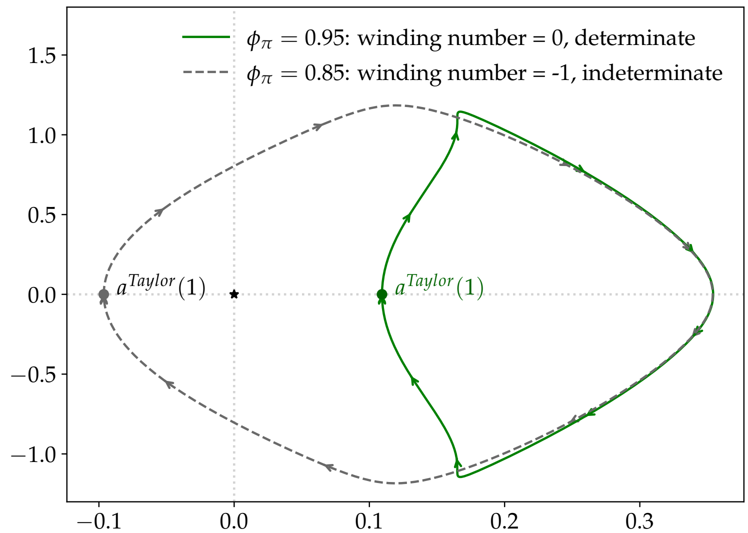


Figure 7: Graph of $a^{Taylor}(z)$

To investigate why the winding number switches from 0 to -1 for high κ and low ϕ_π, we fix κ at 0.05, and vary ϕ_π in figure 7. For each value of ϕ_π, we plot the graph of the symbol $a^{Taylor}(z)$ of $\mathbf{A}^{Taylor}$. We see that, once more, the decisive point that turns the winding number negative is $a^{Taylor}(1)$.

We can use this observation to derive an explicit formula for the Taylor principle in our HA economy. Using the fact that the symbol preserves multiplication and addition of quasi-Toeplitz matrices (see section 3.1), we find that

$$a^{Taylor}(1) = a^Y(1) + a^r(1)\,(\phi_\pi - 1)\,\frac{\kappa}{1-\beta}$$

Here, $a^Y(1)$ is the steady state sensitivity of asset demand to after tax income. $a^r(1)$ is the steady state sensitivity of asset demand to real interest rates. As figure 7 suggests, determinacy is ensured precisely when the symbol evaluated at $z = 1$ is positive, that is, $a^{Taylor}(1) > 0$. Rearranging, we find

$$\phi_\pi > 1 - \frac{1-\beta}{\kappa}\frac{a^Y(1)}{a^r(1)}$$

Substituting in (23), this further simplifies to

$$\phi_\pi > 1 - \frac{1-\beta}{\kappa}\frac{B_{ss}}{Y_{ss} - T_{ss}}\frac{1}{a^r(1)} \tag{28}$$

This heterogeneous-agent version of the Taylor principle exactly illustrates under which conditions the economy is determinate. Crucially, as either the Phillips curve slope parameter κ or the steady state sensitivity of asset demand to interest rates $a^r(1)$ becomes large, the heterogeneous-agent Taylor principle (28) converges back to the regular one, $\phi_\pi > 1$. For small κ and $a^r(1)$, however, the condition (28) can deviate from $\phi_\pi > 1$ considerably (see figure 6).

Note that, while it is clear that the winding number falls by 1 when ϕ_π falls below the right hand side in this condition, we cannot show that the formula (28) is always the correct Taylor principle. We have been unsuccessful in finding a parameterization of the HA model where the formula does not hold, however.

5 Practical Considerations

We end by discussing several practical considerations. Subsection 5.1 explains how the winding number can be computed in practice; we lay out a method to check the "genericity" in proposition 1 in subsection 5.2; and finally in subsections 5.3 and 5.4, we explore how the regions of indeterminacy and feasibility can be computed should the winding number of $\mathbf{J}$ in (2) not be equal to 0.

5.1 Computing the winding number

Suppose that we have a truncated version $\mathbf{j} = \{j_k\}_{k=-\tau}^{\tau}$ of the two-sided sequence $\{j_k\}_{-\infty}^{\infty}$ for some quasi-Toeplitz Jacobian $\mathbf{J}$. Using this truncated $\mathbf{j}$, we have the symbol

$$j(z) = \sum_{k=-\tau}^{\tau} j_k z^k \tag{29}$$

and then evaluate the winding number from definition 3: how many times the graph of $j(z)$ rotates counterclockwise around zero as z goes counterclockwise around $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

To do so in practice, we evaluate $j(z)$ using (29) for a very large, even number N of roots of unity $z = \omega^n$, $n = 0, \dots, N-1$, where $\omega \equiv e^{-2\pi i/N}$. This can be done efficiently using the fast Fourier transform as follows.[14] Define the sequence $\{x_l\}_{l=0}^{N-1}$ by $x_l \equiv j_{l-N/2}$ for $|l - N/2| \leq \tau$ and $x_l \equiv 0$ otherwise. This sequence has the entries of $\{j_k\}$ at its center, padded with zeros on both sides. Then write:

$$\begin{aligned} j(\omega^n) &= \sum_{k=-\tau}^{\tau} j_k e^{-\frac{2\pi i}{N}kn} = \sum_{k=-\tau}^{\tau} x_{k+N/2} e^{-\frac{2\pi i}{N}kn} \\ &= \sum_{l=0}^{N-1} x_l e^{-\frac{2\pi i}{N}\left(l+\frac{N}{2}\right)n} = (-1)^n \sum_{l=0}^{N-1} x_l e^{-\frac{2\pi i}{N}ln} \end{aligned} \tag{30}$$

The rightmost sum in (29) is simply the nth output of the discrete Fourier transform of $\{x_l\}$; all N outputs can be rapidly calculated using the fast Fourier transform, in $O(N \log N)$ time. Hence, to

[14] The following method requires at least $N > 2\tau$, although typically we want a much larger N.

obtain all $j(\omega^n)$, all we need is to form the padded sequence $\{x_l\}$, take the fast Fourier transform, and multiply the results by $(-1)^n$.

The points $z = 1, \omega^{N-1}, \dots, \omega$ move counterclockwise around the unit circle $\mathbb{T}$, and for high N, the piecewise linear curve formed by connecting the points $j(1), j(\omega^{N-1}), \dots, j(\omega)$ closely approximates the graph of $j(z)$ as z goes counterclockwise around $\mathbb{T}$. We can calculate how many times it rotates counterclockwise around zero by counting the number of counterclockwise crossings of the right real axis $\{(x,0) : x \geq 0\}$ (counting negatively any clockwise crossings).[15]

To test the accuracy of this calculation, one can vary N to make sure it is insensitive (we usually start with $N = 2^{13}$). Another useful diagnostic is to plot the graph of $j(1), j(\omega^{N-1}), \dots, j(\omega)$, as we have done in many figures above, and to see whether there are any near-crossings, in which case a higher N is appropriate.

Obtaining j. Above, we saw how to compute the winding number from $\mathbf{j} = \{j_k\}_{k=-\tau}^{\tau}$, a truncated version of the two-sided sequence underlying a quasi-Toeplitz Jacobian. How can we obtain $\mathbf{j}$?

In general, supposing that we have a large truncated $T \times T$ version of $\mathbf{J}$, calculated using the methods from Auclert et al. (2021), we can choose some τ such that $\tau \gg 0$ and $\tau \ll T$, and then write:

$$j_k = \begin{cases} J_{\tau+k,\tau} & -\tau \leq k \leq 0 \\ J_{\tau,\tau-k} & 0 \leq k \leq \tau \end{cases}$$

This extracts $\{j_k\}$ from the final row and column of the leading $(\tau+1) \times (\tau+1)$ submatrix of $\mathbf{J}$. It is accurate to the extent that the compact correction $\mathbf{E}$ is approximately zero in the τth row and column.[16] One way to test this is to evaluate whether $J_{t,s} \approx J_{t-1,s-1}$ for all t. One should also test whether τ is high enough by seeing that j_k is close to zero near $k = -\tau$ and $k = \tau$.

We can sometimes achieve greater accuracy by using the structure of the model. For instance, if $\mathbf{J}$ is the Jacobian of a heterogeneous-agent block as in section 3.3, we can calculate $\{j_k\}$ directly using (17). If it is the Jacobian of a simple block as in section 3.2, $\{j_k\}$ is immediate. Finally, if it is calculated by adding and multiplying Jacobians, we can evaluate the symbol $j(z)$ at any point z by calculating the symbols at z for the underlying Jacobians, then adding and multiplying the results—since, as discussed in section 3.1, the symbol of a sum (or product) is the sum (or product) of the symbols.

[15] To determine whether a line segment from (x_{j-1}, y_{j-1}) to (x_j, y_j) crosses the right real axis, we first test whether the signs of y_{j-1} and y_j are different (counting 0 as positive). Then if x_{j-1} and x_j are both positive, there is an unambiguous crossing; if they are both negative, there is not a crossing. If x_{j-1} and x_j have different signs, then the line segment crosses the right real axis if $\frac{x_{i-1}y_i - x_i y_{i-1}}{y_i - y_{i-1}} > 0$. A crossing is counterclockwise if $y_j > y_{j-1}$ and clockwise otherwise.

[16] In principle, if $\mathbf{J}$ is a truncated version of the true quasi-Toeplitz operator, then $\mathbf{E}$ is likely to be smallest when $\tau = T - 1$, so that we use the last row and column. But when $\mathbf{J}$ is formed in part by composing or inverting truncated matrices, as in the general method of Auclert et al. (2021), artifacts will appear near the point of truncation, and to avoid these artifacts it is best to choose $\tau \ll T$.

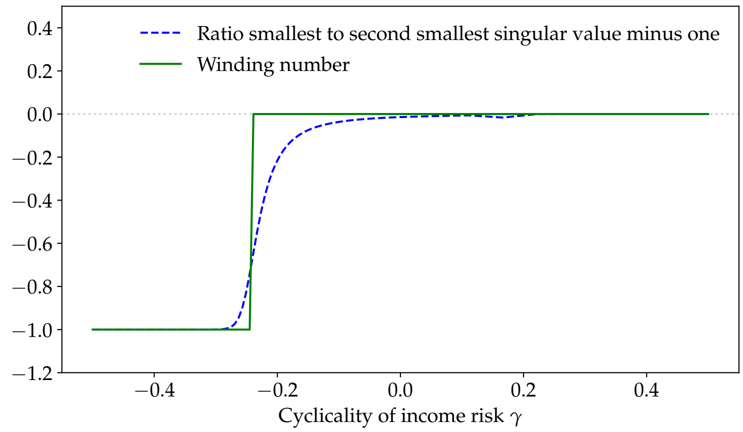


Figure 8: Winding number vs smallest singular value

Note. For the SVD we use the asset Jacobian $\mathbf{A}^{HA}$ truncated to be of size 1001×1001.

5.2 Checking genericity with zero winding number

Our main result is that a quasi-Toeplitz operator $\mathbf{J}$ with zero winding number is generically invertible. What happens in the non-generic case, where $\mathbf{J}$ is not invertible? Given (5), (6) and the fact that it has a zero winding number, $\mathbf{J}$ must be neither surjective nor injective in this case.

Writing

$$\mathbf{J} = \mathbf{T}(\mathbf{j}) + \mathbf{E} = \mathbf{T}(\mathbf{j}) \left(\mathbf{I} + \mathbf{T}(\mathbf{j})^{-1} \mathbf{E} \right)$$

we still have that the exactly Toeplitz component $\mathbf{T}(\mathbf{j})$ is invertible (see the proof of proposition 1). This means $\mathbf{I} + \mathbf{T}(\mathbf{j})^{-1}\mathbf{E}$ is no longer invertible. Mathematically, $\mathbf{T}(\mathbf{j})^{-1}\mathbf{E}$ must have an eigenvalue of -1. This gives a simple approach to check genericity: Compute a truncated version of $\mathbf{T}(\mathbf{j})^{-1}\mathbf{E}$ and see if it has an eigenvalue that is close to -1 and moves closer to -1 as the truncation horizon is increased.

A second, more general approach one can take is that one can compute a singular value decomposition (SVD) of a truncated version of the jacobian $\mathbf{J}$ and see if the smallest singular value is close to zero. In practice, we recommend computing the ratio of the smallest singular value to the second smallest and then checking whether that is below some small threshold.

We illustrate this in figure 8. We compute the ratio of the smallest singular value to the second smallest as well as the winding number of the heterogeneous-agent asset Jacobian $\mathbf{A}^{HA}$ as function of the cyclicality of income risk γ. The figure clearly shows that they both line up well.

Both the eigenvalue approach an the SVD approach work well for detecting genericity issues related to the compact correction as they are, one way or another, both based on truncated representations of $\mathbf{J}$. The winding number focuses on invertibility more generally, without relying on the behavior of $\mathbf{J}$ on any finite-dimensional subspace.

5.3 Computing alternative solutions under indeterminacy

If the Jacobian $\mathbf{J}$ has a negative winding number, it can be interesting to determine the range of solutions $\mathbf{x}$ to (2) that is possible, i.e. the kernel of $\mathbf{J}$. The singular value decomposition allows us to do this too. In fact, the right singular vector associated with the smallest singular value of $\mathbf{J}$ exactly characterizes the dimension of indeterminacy of the solution. More precisely, in most cases we can evaluate a specific solution $\mathbf{x}$ to (2) by inverting the truncated Jacobian $\mathbf{J}^{(T\times T)}$,

$$\mathbf{x}^{(T)} = \left(\mathbf{J}^{(T\times T)}\right)^{-1} \mathbf{y}^{(T)}$$

Denoting by $\mathbf{x}^{r}$ the right singular vector associated with the smallest singular value of $\mathbf{J}^{(T\times T)}$, the space of solutions is then given by

$$\mathbf{x}^{(T)} + \lambda \mathbf{x}^{(r)} \qquad \text{for } \lambda \in \mathbb{R}$$

5.4 Computing the feasible region under non-existence

Vice versa, if we are trying to evaluate the infeasible region for $\mathbf{y}$ for which there does not exist a solution to (2), i.e. the cokernel of $\mathbf{J}$, we consider the left singular vector $\mathbf{x}^{(l)}$ associated with the smallest singular value of the truncated Jacobian $\mathbf{J}^{(T\times T)}$. Then, the subspace of ℓ^2 for which (2) has a solution is the hyperplane orthogonal to $\mathbf{x}^{(l)}$, i.e.

$$\{\mathbf{y} \in \ell^2 : \sum_{t=0}^{\infty} y_t x_t^{(l)} = 0\}$$

6 Conclusion

This note develops a determinacy and existence criterion for models in the sequence space. The criterion is very useful in the budding literature that analyzes heterogeneous-agent models in the sequence space, including heterogeneous-agent New-Keynesian models. We use the criterion to derive a general Taylor principle for those models. In future work, we plan on extending our results to the case where within each time period, the unknown x_t is multi-dimensional.

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A Three Helpful Lemmas

Lemma 2. *Let $\mathcal{C}$ be the space of compact operators on ℓ^2 endowed with the operator norm. Let $\tilde{\mathcal{C}}$ be the set of compact operators for which $\mathbf{I} + \mathbf{E}$ is invertible. Then, $\tilde{\mathcal{C}}$ is open and dense in $\mathcal{C}$.*

Proof. We first show that $\tilde{\mathcal{C}}$ is open. Since $\mathcal{C}$ is closed (Rudin 1991, Theorem 4.18(c)), it is sufficient to show that $\mathcal{C} \backslash \tilde{\mathcal{C}}$ is closed. Let $\mathbf{E}^{(n)} \in \mathcal{C} \backslash \tilde{\mathcal{C}}$ be a convergent sequence of compact operators with limit $\mathbf{E}$ (in the operator norm) for which $\mathbf{I} + \mathbf{E}^{(n)}$ is not invertible. This means, $\mathbf{E}^{(n)}$ has an eigenvalue of -1 for all n. By Dunford and Schwartz (1988, part II, XI-9, Lemma 5), this must mean that $\mathbf{E}$ also has an eigenvalue of -1 and thus that $\mathbf{I} + \mathbf{E}$ is not invertible either. This establishes that $\mathcal{C} \backslash \tilde{\mathcal{C}}$ is closed.

Next, we show that $\tilde{\mathcal{C}}$ is dense in $\mathcal{C}$. For this, take an operator $\mathbf{E} \in \mathcal{C} \backslash \tilde{\mathcal{C}}$ for which $\mathbf{I} + \mathbf{E}$ is not invertible and pick an arbitrary $\epsilon > 0$. We need to show that there exists an operator $\mathbf{E}' \in \tilde{\mathcal{C}}$ for which $\mathbf{I} + \mathbf{E}'$ *is* invertible and that is close to $\mathbf{E}$, $\|\mathbf{E} - \mathbf{E}'\| < \epsilon$. Since $\mathbf{I} + \mathbf{E}$ is not invertible, -1 is an eigenvalue of $\mathbf{E}$. We know from Rudin (1991, Theorem 4.24(b)) that the spectrum of a compact operator is at most countably infinite. This means that there exists a $\delta < \frac{\epsilon}{\|\mathbf{E}\|+1}$ such that $-\frac{1}{1+\delta}$ is *not* an eigenvalue of $\mathbf{E}$, or in other words, that $\mathbf{I} + (1+\delta)\,\mathbf{E}$ is invertible. Clearly then, $\mathbf{E}' \equiv (1+\delta)\,\mathbf{E} \in \tilde{\mathcal{C}}$ and also $\|\mathbf{E}' - \mathbf{E}\| = \delta \|\mathbf{E}\| < \epsilon$. This proves that $\tilde{\mathcal{C}}$ is dense in $\mathcal{C}$ and thus lemma 2. □

Lemma 3. *If $\mathbf{T}$ is a surjective operator, left multiplication with $\mathbf{T}$ is a surjective continuous map from $\mathcal{C}$ to $\mathcal{C}$. If $\mathbf{T}$ is an injective operator with a closed range, right multiplication with $\mathbf{T}$ is also a surjective continuous map from $\mathcal{C}$ to $\mathcal{C}$.*

Proof. Left multiplication with any bounded operator is continuous because for any other bounded operator $\mathbf{E}$ we have

$$\|\mathbf{TE}\| \leq \|\mathbf{T}\| \cdot \|\mathbf{E}\|.$$

Likewise, right multiplication with any bounded operator is continuous.

Let $\mathbf{T}$ now be a surjective (bounded) operator. To show that left multiplication is surjective as map from $\mathcal{C}$ to $\mathcal{C}$, let $\mathbf{E} \in \mathcal{C}$ be a compact operator. By Axler (2020, Theorem 10.31), $\mathbf{T}$ is right invertible, that is, there exists a bounded operator $\mathbf{S}$ such that $\mathbf{TS} = \mathbf{I}$. Defining $\mathbf{E}' \equiv \mathbf{SE}$ we see that $\mathbf{TE}' = \mathbf{E}$. Thus, left multiplication with $\mathbf{T}$ is a surjective continuous map from $\mathcal{C}$ to $\mathcal{C}$.

Let $\mathbf{T}$ be an injective (bounded) operator with a closed range. To show that right multiplication is surjective as map from $\mathcal{C}$ to $\mathcal{C}$, we let $\mathbf{E} \in \mathcal{C}$ be a compact operator. By Axler (2020, Theorem 10.29), $\mathbf{T}$ is left invertible, that is, there exists a bounded operator $\mathbf{S}$ such that $\mathbf{ST} = \mathbf{I}$. Defining $\mathbf{E}' \equiv \mathbf{ES}$ we immediately obtain $\mathbf{E}'\mathbf{T} = \mathbf{E}$. Thus, right multiplication with $\mathbf{T}$ is a surjective continuous map from $\mathcal{C}$ to $\mathcal{C}$. □

Lemma 4. *For any open and dense subset $\hat{\mathcal{C}}$ of $\mathcal{C}$, $\mathbf{T} \cdot \hat{\mathcal{C}} \equiv \{\mathbf{T} \cdot \mathbf{E} | \mathbf{E} \in \hat{\mathcal{C}}\}$ is also open and dense in $\mathcal{C}$ for any surjective operator $\mathbf{T}$; and $\hat{\mathcal{C}} \cdot \mathbf{T} \equiv \{\mathbf{E} \cdot \mathbf{T} | \mathbf{E} \in \hat{\mathcal{C}}\}$ is open and dense in $\mathcal{C}$ for any injective operator $\mathbf{T}$.*

Proof. Right and left multiplication with any bounded operator **T** is continuous and preserves compactness, that is, it maps $\mathcal{C}$ into itself (Rudin 1991, Theorem 4.18(f)). Thus, right and left multiplication map any open subset of $\mathcal{C}$ into another open subset of $\mathcal{C}$.

- If **T** is also surjective, left-multiplication with **T** maps any dense subset of $\mathcal{C}$ into another dense subset of $\mathcal{C}$. This follows straight from left multiplication being continuous and surjective in this case (lemma 3). Continuous surjective functions map dense subsets into dense subsets.

- If **T** is also injective, right-multiplication with **T** maps any dense subset of $\mathcal{C}$ into another dense subset of $\mathcal{C}$. This follows straight from right multiplication being continuous and surjective in this case (lemma 3). The rest of the argument is the same.

This proves lemma 4. □