

# Unique equilibrium in the Eaton-Gersovitz model of sovereign debt

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## Abstract

A common view of sovereign debt markets is that they are prone to multiple equilibria. We prove that, to the contrary, Markov perfect equilibrium is unique in the widely studied model of [Eaton and Gersovitz \(1981\)](#), and we discuss multiple extensions and limitations of this finding. Our results show that no improvement in a borrower's reputation for repayment can be self-sustaining, thereby strengthening the [Bulow and Rogoff \(1989\)](#) argument that debt cannot be sustained by reputation alone.

**Keywords:** Sovereign Debt, Default, Multiple Equilibria

**JEL Codes:** D52, E44, F34, H63

## 1 Introduction

A common view of sovereign debt markets is that they are prone to multiple equilibria: a market panic may inflate bond yields, deteriorate the sustainability of government debt and precipitate a default event, justifying investor fears. Indeed, Mario Draghi's speech in July 2012, announcing that the ECB was "ready to do whatever it takes" to preserve the single currency, and the subsequent creation of the Outright Monetary Transactions (OMT) program, are widely seen as having moved Eurozone sovereign debt markets out of an adverse equilibrium: since then, bond spreads have experienced dramatic falls as fears of default have receded.

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At the same time, in the last decade, a booming quantitative literature in the line of [Eaton and Gersovitz \(1981\)](#)—initiated by [Arellano \(2008\)](#) and [Aguiar and Gopinath \(2006\)](#), and summarized in [Aguiar and Amador \(2015\)](#)—has studied sovereign debt markets using an infinite-horizon incomplete markets model for which no result on equilibrium multiplicity was known. Many researchers suspected that the model might feature multiple equilibria, as discussed by [Hatchondo, Martinez and Sapriz \(2009\)](#):

[Krusell and Smith \(2003\)](#) show that, typically, there is a problem of indeterminacy of Markov-perfect equilibria in an infinite-horizon economy. In order to avoid this problem, we analyze the equilibrium that arises as the limit of the finite-horizon economy equilibrium.

On the other hand, in numerical computations, the literature had not found any explicit case of multiplicity. In this paper we explain why, by proving that equilibrium is *unique* in the benchmark infinite-horizon model with a Markov process for the exogenous driving state and exogenous value from default. Although we emphasize Markov perfect equilibrium—the usual equilibrium concept in the literature, and one for which our argument is especially direct—we show that our core uniqueness result extends to subgame perfect equilibria more generally. We also extend our proof to several modifications of the benchmark model, as described below.

Why could multiplicity arise in the benchmark model we study? To build intuition, consider the simplest environment: one in which debt is restricted to be risk-free, as in [Zhang \(1997\)](#). A Markov perfect equilibrium of this model features a (constant) endogenous debt limit, which is the most that can be incurred today without the possibility that the government will want to default tomorrow. Suppose credit becomes tighter in the future—tomorrow’s debt limit falls. Since the government is less able to smooth consumption fluctuations, its perceived benefit from access to credit is now lower, and so its willingness to repay today’s debts falls. In response, investors lower today’s debt limit as well.

Through this process, an equilibrium with loose credit and high willingness to repay debts could turn into one with tight credit and low willingness to repay. Similarly, in the full [Eaton and Gersovitz \(1981\)](#) model with risky debt, investors’ pessimistic expectations about the likelihood of default could translate into higher risk premia on debt—which, by making debt service more costly and continued access to credit markets less valuable, would encourage default and validate the original pessimism. This mechanism sounds appealing, and in our view it captures an important part of the common intuition for equilibrium multiplicity in sovereign debt markets.

Our results rule it out. The intuition remains simplest in the [Zhang \(1997\)](#) environment. If there are two equilibria with distinct debt limits, we consider two governments that are each at the limit in their respective equilibria. We argue that the government with less debt must have a strictly higher value: starting from that point, it can follow a strategy that parallels the strategy of the higher-debt government, maintaining its liabilities at a uniform distance and achieving higher consumption at every point by economizing on interest payments. But this contradicts the assumption that both governments start at their debt limits, where each must obtain the (constant)

value of default. In short, once both governments have exhausted their debt capacity, the one with a strictly lower level of debt is strictly better off—meaning that this government should be able to borrow slightly more without running the risk of default, and cannot have exhausted its capacity after all.

Interestingly, this proof strategy by replication has echoes of that used by [Bulow and Rogoff \(1989\)](#) to rule out reputational equilibria in a similar class of models where sovereign governments retain the ability to save after defaulting. The original Bulow-Rogoff result is cast in a complete markets setting. In a second modification of the benchmark model, we specify the only punishment from default as the loss of ability to borrow. As an immediate corollary to our Eaton-Gersovitz uniqueness result, we then obtain the *incomplete markets Bulow-Rogoff result*: under this specification of default costs, the no-borrowing equilibrium is the unique equilibrium. Hence our general uniqueness result nests a key impossibility result for the sovereign debt literature. Here, our paper complements parallel and independent work by [Bloise, Polemarchakis and Vailakis \(2016\)](#), who explore the validity of the Bulow-Rogoff result in environments with general asset market structures.

We next explore the robustness of our uniqueness result to relaxing various model assumptions. We first consider a case where savings are exogenously bounded. Echoing a result of [Pasadore and Xandri \(2014\)](#), we prove that multiple equilibria can exist when no savings is allowed. We also show, however, that uniqueness holds whenever the bound on savings is strictly positive. Next, we consider a case where the value of default is endogenous because governments in default have a stochastic option to reenter markets (a typical assumption in the quantitative literature). In that case, we rule out multiplicity of the most widely suspected form—where bond prices in a favorable equilibrium dominate those in a self-fulfilling adverse one—and obtain complete uniqueness when shocks are independent and identically distributed. Finally, we discuss alternative assumptions that are known to generate multiple equilibria in related contexts, including modifying the timing and commitment assumptions, introducing long-term debt, or assuming low international interest rates.

Our results are important because they show that the multiplicity intuition is *not* valid in a benchmark model that is accepted as a good description—both qualitative and quantitative—of sovereign debt markets.<sup>1</sup> They provide fresh analytical insight for a model that has few theoretical results to date, making use of a powerful new proof technique along the way. And they show that alternative strategies to compute Markov perfect equilibria should all converge to the same solution. Our results are not directly applicable to all the extensions of the Eaton-Gersovitz model that the quantitative sovereign debt literature has considered, but they do suggest that multiplicity is unlikely in many of these cases as well, and therefore that the literature’s quantitative findings

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<sup>1</sup>In particular, under our assumptions, sunspots cannot influence equilibrium outcomes. Recently, [Stangebye \(2015\)](#) has explored the role of sunspots in two versions of the Eaton-Gersovitz model where our results do not apply—first, for short-term debt, when the domain of debt is exogenously restricted beyond what we consider in section 4.1, and second, for long-term debt.

are probably not driven by a hidden equilibrium selection.<sup>2</sup>

Our objective is not to deny that sovereign debt markets can be prone to self-fulfilling crises, or that OMT may have ruled out a bad equilibrium. Instead, we hope that our results may help sharpen the literature’s understanding of the assumptions that are needed for such multiple equilibria to exist. Our replication-based proof strategy may also be of independent interest, as a general technique for proving uniqueness of equilibrium in infinite-horizon games.

**Layout** The rest of the paper is organized as follows. Section 2 lays out the benchmark Eaton-Gersovitz model, and establishes uniqueness of Markov perfect equilibrium and uniqueness of subgame perfect equilibrium. Section 3 adapts our main proof to two related models. Section 3.1 proves uniqueness in the Zhang (1997) model, where debt is restricted to be risk-free, and section 3.2 derives the incomplete markets version of the Bulow and Rogoff (1989) result as a corollary of our main uniqueness result. Section 4 considers the robustness of our results as we relax various assumptions. Section 4.1 considers exogenous restrictions on savings. Section 4.2 considers the case where reentry is allowed after default. Section 4.3 discusses other extensions. Section 5 concludes. Proofs not included in the main text are collected in the appendix.

## 2 Equilibrium uniqueness in the benchmark model

In this section we describe our benchmark environment, provide a proof of existence, and move on to establish the core uniqueness result of the paper.

### 2.1 Model description

We now describe what we call the *benchmark* infinite-horizon model with Markov income (see Aguiar and Amador 2015). We focus first on Markov perfect equilibria, in which the current states  $b$  and  $s$  encode all the relevant history. In section 2.3 we will show that this is without loss of generality, since one can specify this model as a game whose only subgame perfect equilibria are Markov perfect equilibria.

An exogenous state  $s$  follows a discrete Markov chain with elements in  $\mathcal{S}$ ,  $|\mathcal{S}| = S \in \mathbb{N}$  and transition matrix  $\pi(s'|s)$ . Output  $y(s)$  is a function of this underlying state.

At the beginning of each period, the government starts with some level of debt  $b \in \mathcal{B}$ . After observing the realization of  $s$ , it decides whether to repay  $b$  or default. If it does not repay, it receives an exogenous value  $V^d(s)$ , which encodes all the consequences of default. For example, if default is punished by permanent autarky, with output also reduced by an exogenous cost  $\tau(s) \in$

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<sup>2</sup>While our focus is on sovereign debt, the benchmark model we study also constitutes the core of a literature that analyzes unsecured consumer credit (Chatterjee, Corbae, Nakajima and Ríos-Rull, 2007), and we conjecture that equilibrium is also unique in many of the models used in that literature.

$[0, y(s)]$ , then  $V^d$  is defined recursively by

$$V^d(s) = u(y(s) - \tau(s), s) + \beta \mathbb{E}_{s'|s} [V^d(s')] \quad (1)$$

The assumption that  $V^d(s)$  is exogenous and independent of  $b$  follows some of the quantitative sovereign debt literature. This assumption is important for the current result. When the literature has considered endogenous  $V^d(s)$ , it has typically been by including a stochastic reentry option; we will consider this possibility in section 4.2.

If the government does not default, it receives  $y(s)$  as endowment, pays  $b$ , and issues new bonds  $b' \in \mathcal{B}$  that will be due next period, raising revenue  $Q(b', s)$ . Its (possibly state-dependent) flow utility from consumption is  $u(c, s)$ , so that the value  $V$  from repayment is given by

$$\begin{aligned} V(b, s) &= \max_{b'} u(c, s) + \beta \mathbb{E}_{s'|s} [V^o(b', s')] \\ \text{s.t. } &c + b = y(s) + Q(b', s) \end{aligned} \quad (2)$$

and the value  $V^o$  including the option to default at the beginning of a period is given by

$$V^o(b, s) = \max_{p \in \{0,1\}} pV(b, s) + (1-p)V^d(s) \quad (3)$$

where  $p = 1$  denotes the decision to repay and  $p = 0$  denotes the decision to default.

Debt is purchased by risk-neutral international investors that demand an expected return of  $R$ . For convenience, we assume that when a government is indifferent between repayment and default, it chooses to repay:  $p(b, s) = 1$  if and only if  $V(b, s) \geq V^d(s)$ . Since investors receive expected repayment  $\mathbb{E}_{s'|s}[p(b', s')]$ , it follows that the bond revenue schedule  $Q$  is

$$Q(b', s) = \frac{b'}{R} \mathbb{E}_{s'|s} [p(b', s')] = \frac{b'}{R} \mathbb{P}_{s'|s} [V(b', s') \geq V^d(s')] \quad (4)$$

We are now ready to define Markov perfect equilibrium, which is the typical focus in the literature.

*Definition 1.* A Markov perfect equilibrium is a set of policy functions  $p(b, s)$ ,  $c(b, s)$ ,  $b'(b, s)$  for repayment, consumption and next period debt, value functions  $V(b, s)$  and  $V^o(b, s)$ , and a bond revenue schedule  $Q(b', s)$ , all defined on the set  $\mathcal{B} \times \mathcal{S}$ , such that (2)-(4) are satisfied.

We first prove an existence result—to our knowledge, the first such formal result in the literature. For this we make the following four assumptions.

*Assumption 1.*  $\beta \in (0, 1)$ , and for each  $s$ ,  $u(\cdot, s)$  is continuous and strictly increasing.

*Assumption 2.* There exist  $\gamma > 0$  and  $\kappa > 0$  such that  $u(c, s) \leq \gamma c^\kappa$  for all  $c, s$ ; and  $\beta R^\kappa < 1$ .

*Assumption 3.*  $\lim_{c \rightarrow 0} u(c, s) = -\infty$ .

*Assumption 4.*  $\mathcal{B} = [\underline{b}, \bar{b}]$ , where  $-\infty \leq \underline{b} \leq 0 < \bar{b} < \infty$ .

Assumption 1 is a standard restriction on preferences. Assumption 2 guarantees that the government cannot obtain unboundedly high utility by deferring consumption indefinitely and earn-

ing interest on the resulting savings. Assumption 3 ensures that the government is never at a corner of zero consumption, since such corner solutions can lead to analytically intractable discontinuities in  $V$ . Assumptions 2 and 3 are jointly satisfied by some common parametric specifications, including CRRA utility  $u(c, s) = \frac{c^{1-\sigma}-1}{1-\sigma}$  for any  $\sigma \geq 1$ .

Assumption 4 includes several restrictions on allowable bond positions. The upper bound  $\bar{b} < \infty$  rules out Ponzi schemes; it can be chosen high enough not to be binding.  $\bar{b} > 0$  restricts our focus to cases where debt is allowed. Our assumption that  $\underline{b} \leq 0$  allows the government to pay down all of its debt if it so desires, and possibly to save.<sup>3</sup> For now,  $\underline{b}$  is left otherwise unrestricted. (Later, in sections 2.2 and 4.1, we will establish separate uniqueness results for the cases of  $\underline{b} = -\infty$  and  $\underline{b} > -\infty$ , respectively. In the latter case, assumption 2 is superfluous.)

We also need an assumption to guarantee that default is never optimal when the government has positive assets. Define  $V^{nb}(b, s)$  to be the value function for a government that can save at the risk-free rate but not borrow,

$$\begin{aligned} V^{nb}(b, s) &= \max_{b'} u(c, s) + \beta \mathbb{E}_{s'|s} [V^{nb}(b', s')] \\ \text{s.t. } c + b &= y(s) + \frac{b'}{R}, \quad \underline{b} \leq b' \leq 0 \end{aligned} \quad (5)$$

Then we assume

*Assumption 5.*  $-\infty < V^d(s) \leq V^{nb}(0, s)$ .

Assumption 5 is satisfied, for example, when  $V^d$  is given by (1) for any exogenous cost of default  $\tau(s) \geq 0$ .

**Proposition 1.** *Under assumptions 1–5, a Markov perfect equilibrium exists. In any equilibrium,  $V(b, s)$  is strictly decreasing in  $b$  for each  $s$ , and there exists a set of default thresholds  $\{b^*(s)\}_{s \in \mathcal{S}}$  such that the government repays in state  $s$  if and only if  $b \leq b^*(s)$ . Both  $V$  and  $Q$  are uniquely determined by the thresholds  $\{b^*(s)\}_{s \in \mathcal{S}}$ .*

The proof, developed in appendix A.1, is constructive and relies on a fixed-point procedure similar to the one used by the quantitative literature to search for an equilibrium. As highlighted by Aguiar and Amador (2015), this procedure involves iterating on a monotone and bounded operator in the space of default thresholds. These iterations converge to a fixed point, and our proof verifies that this fixed point defines an equilibrium. Our assumptions ensure that value functions exist, are continuous and finite-valued, and that default thresholds  $b^*(s)$  are uniquely defined by the equalities

$$V(b^*(s), s) = V^d(s) \quad (6)$$

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<sup>3</sup>In an environment with  $\underline{b} > 0$ , default frees the government from the otherwise inflexible requirement to borrow at least  $\underline{b}$ . This creates a reward for defaulting which can be difficult to interpret and is not the typical focus of the literature.

The set  $\{b^*(s)\}_{s \in \mathcal{S}}$  then characterizes the bond revenue schedule  $Q$ : following (4),

$$Q(b', s) = \frac{b'}{R} \mathbb{P}_{s'|s} [b' \leq b^*(s')] = \frac{b'}{R} \sum_{\{s': b' \leq b^*(s')\}} \pi(s'|s) \quad (7)$$

In the special case where  $u(c, s) = u(c)$ , income is i.i.d, and  $V^d$  is the expected value of autarky, it is possible to show that  $b^*(s)$  is increasing in  $y(s)$  (see [Arellano 2008](#)), but such monotonicity is not needed for our proof.

In this environment, it is natural to conjecture that multiple equilibria could be present. Starting from an equilibrium with default thresholds  $\{b^*(s)\}_{s \in \mathcal{S}}$ , lowering default thresholds increases the cost of borrowing a given amount  $b'$ , through (7). This, in turn, lowers the value to the government of repaying any amount  $b$ , shifting down the function  $V(\cdot, s)$  for every  $s$ . Through (6), this lowers the levels of debt at which the government is tempted to default, reinforcing the initial impulse in a vicious cycle.

For a given application, one could in principle check such multiplicity directly using a variant of the procedure used in our existence proof. If the iterative procedure, starting from a set of minimal default thresholds, converges to the same fixed point as when starting from a set of maximal default thresholds, it follows from monotonicity that no other equilibrium exists.

In the next section, we provide an alternative argument that establishes uniqueness in every case. Our proof illuminates, in this environment, why the vicious cycle described above is never strong enough to create multiple equilibria, highlighting the key role played by assumptions on  $R$  and  $\underline{b}$ .

## 2.2 Uniqueness of Markov perfect equilibrium

Suppose that we have two distinct revenue schedules  $Q$  and  $\tilde{Q}$ , each derived via (7) from anticipated default thresholds  $\{b^*(s)\}_{s \in \mathcal{S}}$  and  $\{\tilde{b}^*(s)\}_{s \in \mathcal{S}}$ . Let  $V$  and  $\tilde{V}$  be the value functions for a government facing these schedules. To prove uniqueness of equilibrium, we need to show that at most one of these value functions can be consistent with the default thresholds that generate it—in other words, that we cannot have both  $V(b^*(s), s) = V^d(s)$  and  $\tilde{V}(\tilde{b}^*(s), s) = V^d(s)$  for all  $s$ .

The key observation of this paper is that we can derive a simple inequality for the two value functions  $V$  and  $\tilde{V}$ , related to the maximum difference between the default thresholds. This inequality requires assumptions 1–5 together with two crucial new assumptions:

*Assumption 6.*  $R > 1$ .

*Assumption 7.*  $\underline{b} = -\infty$ .

The basis of our inequality is a simple replication strategy we call *mimicking at a distance*. Suppose that  $b^*(s)$  exceeds  $\tilde{b}^*(s)$  by at most  $M > 0$ . Then we show that it is always weakly better to start with debt of  $b - M$  when facing prices  $\tilde{Q}$  than with debt of  $b$  when facing prices  $Q$ , and indeed strictly better whenever  $V(b, s) \geq V^d(s)$ . This observation, formalized in lemma 2, will ultimately be the basis of the proof that distinct equilibria are impossible in proposition 3.



The argument is as follows. The government with debt  $b - M$  facing prices  $\tilde{Q}$  has the option to *mimic* the policy of the government with debt  $b$  facing prices  $Q$ —always defaulting at the same points, and otherwise choosing the same level of debt for the next period minus  $M$ . Such mimicking is possible, irrespective of the value of  $M > 0$ , because savings is unrestricted (assumption 7). Before it defaults, this government is better off because it pays less to service debt, allowing it to consume more.

Debt service, in turn, costs less for two reasons. First, the mimicking government is less likely to be above the default thresholds assumed by its revenue schedule. This is due to the choice of  $M$ : since  $M$  is the maximum amount by which the default thresholds  $b^*(s)$  exceed the thresholds  $\tilde{b}^*(s)$ , as long as the government facing  $\tilde{Q}$  chooses debt of  $M$  less than the government it is mimicking, it is weakly less likely to exceed  $\tilde{b}^*(s)$  than the other government is to exceed  $b^*(s)$ . Second, the mimicking government has strictly less debt, meaning that the cost of servicing this debt is lower, since  $R > 1$  by assumption 6.<sup>4</sup>

Following this policy, the mimicking government always consumes strictly more until default, implying strictly higher utility due to assumption 1. It thus obtains a weakly higher value, which is strictly higher as long as it does not default right away.

**Lemma 2** (Mimicking at a distance.). *Let  $Q$  and  $\tilde{Q}$  be two distinct revenue schedules, with  $Q$  reflecting expected default thresholds  $\{b^*(s)\}_{s \in \mathcal{S}}$  and  $\tilde{Q}$  reflecting expected default thresholds  $\{\tilde{b}^*(s)\}_{s \in \mathcal{S}}$ . Let  $V$  and  $\tilde{V}$  be the respective value functions for governments facing these revenue schedules. Define*

$$M = \max_s b^*(s) - \tilde{b}^*(s) \quad (8)$$

and assume without loss of generality that  $M > 0$ . Then, for any  $s$  and  $b$ ,

$$\tilde{V}(b - M, s) \geq V(b, s) \quad (9)$$

with strict inequality whenever  $V(b, s) \geq V^d(s)$ .

*Proof.* First, note that for any  $b'$  and  $s$ , applying (7) we have

$$\begin{aligned} \tilde{Q}(b' - M, s) &= \frac{(b' - M)}{R} \sum_{\{s': b' - M \leq \tilde{b}^*(s')\}} \pi(s'|s) \geq \frac{(b' - M)}{R} \sum_{\{s': b' \leq b^*(s')\}} \pi(s'|s) \\ &> \left( \frac{b'}{R} \sum_{\{s': b' \leq b^*(s')\}} \pi(s'|s) \right) - M = Q(b', s) - M \end{aligned} \quad (10)$$

Thus the amount that a government with schedule  $\tilde{Q}$  can raise by issuing  $b' - M$  of debt is always strictly larger than the amount that a government with schedule  $Q$  can raise by issuing  $b'$  of debt, minus  $M$ . The two intermediate inequalities in (10) reflect the two sources of this advantage. First, there are weakly more cases in which  $b' - M \leq \tilde{b}^*(s')$  than in which  $b' \leq b^*(s')$ , and this higher

<sup>4</sup>As we will discuss in section 4.3, the  $R > 1$  assumption here plays a crucial role.



chance of repayment makes it possible to raise more. Second, since assumption 6 requires  $R > 1$ , issuing  $M$  less debt costs strictly less than  $M$  in foregone revenue in the current period.

Now we can formally define the *mimicking at a distance* policy. For any states and debt levels  $s$  and  $b$ , let the history  $s^0$  be such that the state and debt owed at  $t = 0$  are respectively  $s$  and  $b$ . The optimal strategy for a government facing schedule  $Q$  induces an allocation  $\{c(s^t), b(s^{t-1}), p(s^t)\}_{s^t \succeq s^0}$  at all histories following  $s^0$ .<sup>5</sup> We construct a policy for the government facing schedule  $\tilde{Q}$  in state  $s$  and debt level  $b - M$  as follows. For every history  $s^t$  following and including  $s^0$ , let

$$\tilde{p}(s^t) = p(s^t)$$

and provided that  $p(s^t) = 1$ , choose consumption and next-period debt as

$$\begin{aligned}\tilde{b}(s^t) &= b(s^t) - M \\ \tilde{c}(s^t) &= c(s^t) + \tilde{Q}(b(s^t) - M, s_t) - (Q(b(s^t), s_t) - M)\end{aligned}\quad (11)$$

Note that  $\tilde{b}(s^t) \in \mathcal{B}$  due to assumption 7. This choice of  $\tilde{b}$  and  $\tilde{c}$  ensures that the budget constraint is satisfied at all histories  $s^t$  where repayment takes place:

$$\begin{aligned}\tilde{c}(s^t) + \tilde{b}(s^{t-1}) - \tilde{Q}(\tilde{b}(s^t), s_t) &= \tilde{c}(s^t) + b(s^{t-1}) - M - \tilde{Q}(b(s^t) - M, s_t) \\ &= c(s^t) + b(s^{t-1}) - Q(b(s^t), s_t) \\ &= y(s_t)\end{aligned}$$

Furthermore, using (10) we see that  $\tilde{c}(s^t) > c(s^t)$ : when there is repayment, the mimicking policy (11) sets consumption  $\tilde{c}(s^t)$  equal to consumption  $c(s^t)$  in the other equilibrium, plus a bonus  $\tilde{Q}(b(s^t) - M, s_t) - (Q(b(s^t), s_t) - M) > 0$  from lower debt costs.

The mimicking policy, of course, need not be optimal; but since it is feasible, it serves as a lower bound for  $\tilde{V}(b - M, s)$ :

$$\begin{aligned}\tilde{V}(b - M, s) &\geq \sum_{\tilde{p}(s^t)=1} \beta^t \Pi(s^t) u(\tilde{c}(s^t), s_t) + \sum_{\tilde{p}(s^t)=0, \tilde{p}(s^{t-1})=1} \beta^t \Pi(s^t) V^d(s_t) \\ &\geq \sum_{p(s^t)=1} \beta^t \Pi(s^t) u(c(s^t), s_t) + \sum_{p(s^t)=0, p(s^{t-1})=1} \beta^t \Pi(s^t) V^d(s_t) = V(b, s)\end{aligned}$$

with strict inequality whenever  $p(s^0) = 1$  (or equivalently  $b \leq b^*(s)$ ), since this implies  $\tilde{c}(s^0) > c(s^0)$  and  $u(c, s_0)$  is strictly increasing in  $c$  thanks to assumption 1.  $\square$

An illustration of the mimicking policy used in lemma 2 is given in figures 1 and 2, which depict time paths in a hypothetical two-state case. In this case, debt starts relatively high and the high-income state  $y(s_H)$  keeps recurring, leading the government to deleverage in anticipation of lower incomes in the future. Figure 1 shows the paths of  $b$  (filled circles) and the mimicking policy

<sup>5</sup> $b(s^t)$  is defined to be the amount of debt chosen at history  $s^t$  to be repaid in period  $t + 1$ .

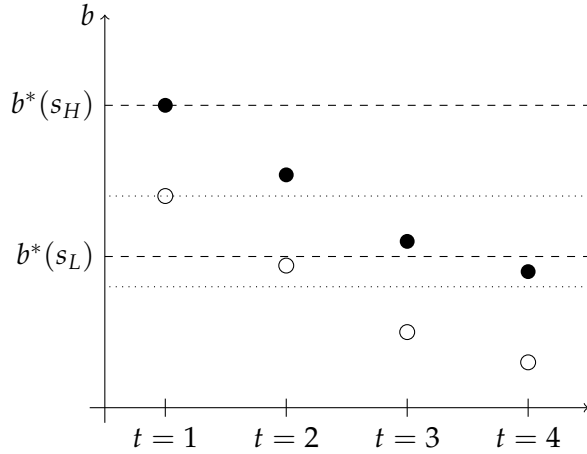


Figure 1: Example paths for  $b$  and  $\tilde{b}$ .

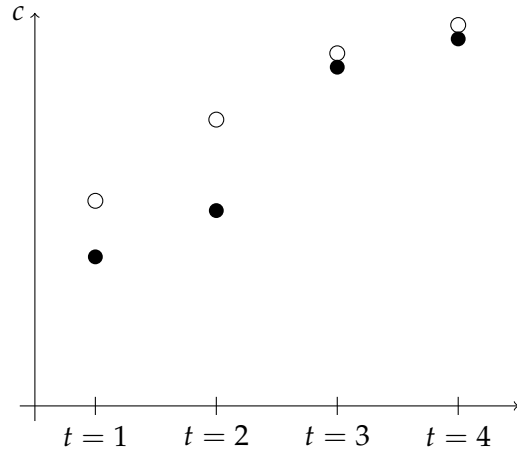


Figure 2: Example paths for  $c$  and  $\tilde{c}$ .

$\tilde{b} = b - M$  (hollow circles), while figure 2 shows the paths of  $c$  (filled circles) and the consumption  $\tilde{c} = c + \tilde{Q}(b - M, s) - (Q(b, s) - M)$  induced by the mimicking policy (hollow circles). Observe that  $\tilde{c}$  is always greater than  $c$ .

The central observation is that if it starts with debt  $M = \max_s b^*(s) - \tilde{b}^*(s)$  below the other government, the mimicking government can keep itself at the fixed distance  $M$ , achieving higher consumption along the way.

We now turn to the main result, which uses lemma 2 to rule out multiple equilibria  $(V, Q)$  and  $(\tilde{V}, \tilde{Q})$  altogether.

**Proposition 3.** *In the benchmark model, Markov perfect equilibrium has a unique value function  $V(b, s)$  and bond revenue schedule  $Q(b, s)$ .*

*Proof.* Suppose to the contrary that there are distinct equilibria  $(V, Q)$  and  $(\tilde{V}, \tilde{Q})$ . Proposition 1 shows that these are characterized by their default thresholds  $\{b^*(s)\}_{s \in \mathcal{S}}$  and  $\{\tilde{b}^*(s)\}_{s \in \mathcal{S}}$ . Therefore, it suffices for us to show that the thresholds are unique.

Without loss of generality, assume that the maximal difference between  $b^*$  and  $\tilde{b}^*$  is positive and is attained in a state  $\bar{s} \in \mathcal{S}$ :

$$\max_s b^*(s) - \tilde{b}^*(s) = b^*(\bar{s}) - \tilde{b}^*(\bar{s}) = M > 0$$

Applying lemma 2 for  $s = \bar{s}$  and  $b = b^*(\bar{s}) = \tilde{b}^*(\bar{s}) + M$ , we know that

$$\tilde{V}(\tilde{b}^*(\bar{s}), \bar{s}) > V(b^*(\bar{s}), \bar{s})$$

But this contradicts the fact that  $b^*(\bar{s})$  and  $\tilde{b}^*(\bar{s})$  are default thresholds, which requires  $\tilde{V}(\tilde{b}^*(\bar{s}), \bar{s}) = V(b^*(\bar{s}), \bar{s}) = V^d(\bar{s})$ . Thus our premise of distinct equilibria cannot stand.  $\square$

The intuitive thrust of lemma 2 and proposition 3 is that distinct bond revenue schedules

cannot both be self-sustaining. No two schedules  $Q$  and  $\tilde{Q}$  can simultaneously rationalize their corresponding default thresholds  $b^*(\bar{s})$  and  $\tilde{b}^*(\bar{s})$  in the state  $\bar{s}$  where these thresholds differ most. Instead, the argument of lemma 2 shows that it is better for a government to start at the lower threshold  $\tilde{b}^*(\bar{s})$  given schedule  $\tilde{Q}$  than to start at the higher threshold  $b^*(\bar{s})$  given schedule  $Q$ ; at this point, any advantages of  $Q$  over  $\tilde{Q}$  are outweighed by the heavier debt burden, and the former government can use a simple mimicking strategy to guarantee itself strictly higher consumption than the latter. It follows that these cannot both be default thresholds, which by definition must be equally desirable, with common value equal to the default value  $V^d(\bar{s})$ .

### 2.3 Uniqueness of subgame perfect equilibrium

The arguments used to prove proposition 3 can be extended to show that this model admits a unique subgame perfect equilibrium. While the Markov perfect concept exogenously restricts equilibrium to depend on a limited set of states, subgame perfect equilibria allow an arbitrary dependence of strategies at time  $t$  on the history  $h^{t-1}$  of past states and actions. The following result shows that the current states  $s$  and  $b$  summarize this dependence, demonstrating that the Markov concept—which has been the focus of much of the quantitative literature—is not restrictive. Proving this formally requires defining the game played by the government and international investors more precisely. Crucially, in this game, the value from government default is still exogenous—endogenizing the default option as part of the game is outside of the scope of this paper (see Kletzer and Wright 2000, Wright 2002 or Krueger and Uhlig 2006 for such an exercise, and our discussion in footnote 12). Here we summarize our result, and relegate the description of the game and the proof to appendix A.2. Let  $V(h^{t-1}, s)$  be the value achieved by a government after history  $h^{t-1}$ , when the current exogenous state is  $s$ . Then the following result holds.

**Proposition 4.** *Consider two subgame perfect equilibria  $A$  and  $B$ . For any  $(b, s)$ , and any histories  $(h_A, h_B)$  such that  $b(h_A) = b(h_B) = b$ , we have  $V_A(h_A, s) = V_B(h_B, s)$ .*

The key to the proof of proposition 4 is to show that, conditional on the exogenous state  $s$ , a government with higher debt must have lower value, independently of the equilibrium that is played or the history of past actions. This in turn relies on another mimicking argument, whereby a government with lower debt can always choose a strategy that ensures it higher consumption and higher future value than its higher-debt counterpart.

## 3 Application to other models

The argument used to prove uniqueness of equilibrium in section 2 is very general and can be used in other contexts, as the following applications illustrate.

### 3.1 Bewley models with endogenous debt limits

Consider a modification of the environment of section 2, in which lenders are restricted to offer a price of  $\frac{1}{R}$  for every unit of debt that they buy. Borrowing must therefore be risk-free: this is the equilibrium defined in Zhang (1997). This restriction can be captured within the framework of the previous section by specifying that the price of non-riskless debt is zero. Instead of (4), the bond revenue schedule becomes

$$Q^z(b', s) = \frac{b'}{R} \mathbf{1} \left[ V^z(b', s') \geq V^d(s') \quad \forall s' | s \right] \quad (12)$$

Define  $\phi(s)$  as the value that satisfies  $V^z(\phi(s), s) = V^d(s)$ , and assume that for all  $s$  and  $s'$ ,  $\pi(s'|s) > 0$ . Then, writing  $\varphi \equiv \min_s \{\phi(s)\}$ , (12) becomes

$$Q^z(b', s) = \frac{b'}{R} \mathbf{1}_{\{b' \leq \varphi\}} \quad (13)$$

In other words, the model is a standard incomplete markets model in the tradition of Bewley (1977), with a debt limit  $\varphi$  determined endogenously by the requirement that the government should never prefer default.

We can immediately prove analogs of lemma 2 and proposition 3 in this new environment.

**Lemma 5.** *Consider two distinct equilibria with value functions  $V$  and  $\tilde{V}$  and debt limits  $\tilde{\varphi} < \varphi$ . Then, letting  $M = \varphi - \tilde{\varphi}$ , for any  $b$  and  $s$  we have*

$$\tilde{V}(b - M, s) \geq V(b, s) \quad (14)$$

with strict inequality whenever  $b \leq \varphi$ .

*Proof.* Same as the proof of lemma 2, with (4) replaced by (13) and inequality (10) becoming

$$\tilde{Q}^z(b' - M, s) = \frac{b' - M}{R} \mathbf{1}_{\{b' - M \leq \tilde{\varphi}\}} = \frac{b' - M}{R} \mathbf{1}_{\{b' \leq \varphi\}} > \frac{b'}{R} \mathbf{1}_{\{b' \leq \varphi\}} - M = Q^z(b', s) - M \quad (15)$$

□

The intuition behind (14) and (15) is well known in this class of environment: an increase in the debt limit is equivalent to a translation of the value function, accompanied by a translation of the income process that reflects the interest costs of debt.<sup>6</sup> Our earlier inequality (10) can be interpreted as a generalization of this result.

**Proposition 6.** *In the model with riskless debt, Markov perfect equilibrium has a unique value function  $V(b, s)$  and debt limit  $\varphi$ .*

*Proof.* Same as the proof of proposition 3, but using lemma 5 rather than lemma 2. □

<sup>6</sup>See, for example, Ljungqvist and Sargent (2012).

As highlighted in the introduction, this particular application illustrates the key intuition behind our main uniqueness result in section 2: a deterioration in the terms of borrowing cannot be self-sustaining in this class of models since, once governments have exhausted their debt capacity, those with less debt are always better off.

### 3.2 Bulow and Rogoff

Our proof is also related to that used by [Bulow and Rogoff \(1989\)](#) to rule out reputational equilibria in sovereign debt models where saving is allowed after default. As originally written, the Bulow-Rogoff result only applies directly to environments with complete markets, but a similar result also holds in the incomplete markets framework we study: if a government can save at a strictly positive net risk-free rate after defaulting, and there are no other exogenous penalties for default, then no debt can be sustained. Though this result has not—to our knowledge—been written formally until now, it has informally motivated the ingredients of modern variations on the Eaton-Gersovitz model, which all specify some exclusion from international markets after default, together with additional costs of default such as output losses.

Recall from (5) that  $V^{nb}$  is the value of a government that is able to save at the risk-free rate but not borrow. The incomplete markets analog of [Bulow and Rogoff \(1989\)](#) corresponds to the special case where  $V^d(s) \equiv V^{nb}(0, s)$ . In other words, when the government defaults, its debt is reset at 0 and it can subsequently save but not borrow.

**Proposition 7** (Incomplete markets Bulow-Rogoff). *In the model with  $V^d(s) = V^{nb}(0, s)$  (i.e. savings after default), no debt can be sustained: in the unique Markov perfect equilibrium, the default thresholds  $b^*(s)$  equal 0 for all  $s$ , and  $Q(b', s) = 0$  for all  $b' \geq 0$ . Hence  $V(b, s) = V^{nb}(b, s)$ .*

The proof in appendix A.3 has two steps: first, it verifies that there exists an equilibrium with no lending, and second, it applies proposition 3 to show that no other equilibrium is possible. In particular, any equilibrium with debt is ruled out.

Going back to the proof of proposition 3, the intuition behind this result is that once a government has already borrowed the maximum amount that can obtain a nonzero price, access to debt markets offers no benefits beyond access to a market for savings. It is impossible to borrow more until some debt is repaid—and rather than repay and reborrow, it is cheaper to default and then run savings up and down in a parallel way, achieving higher consumption by avoiding the costs of debt service. No amount of debt is sustainable: whenever a government has borrowed the maximum, it will default with certainty, and in anticipation creditors will never allow any debt.

This resembles the logic behind the original [Bulow and Rogoff \(1989\)](#) result, which observed that for a reputational debt contract in complete markets, there must always be some state of nature in which a government can default and use the amount demanded for repayment as collateral for a sequence of state-contingent “cash in advance” contracts that deliver strictly higher consumption in every future date and state. The main idea behind their proof carries over to our incomplete markets environment, once the cash in advance contracts are replaced with a sim-

ple, parallel savings strategy. Our contribution here is to show that this result is a special case of a much broader equilibrium uniqueness result: once the existence of a no-debt equilibrium is verified, the Bulow-Rogoff result follows immediately from proposition 3.<sup>7</sup>

## 4 Extensions of the benchmark model

We now discuss some common variants of the benchmark Eaton-Gersovitz model covered in section 2, showing when the uniqueness result does and does not carry through.

### 4.1 Bound on savings

We first consider the case  $\underline{b} > -\infty$ , dropping assumption 7 and replacing it with

*Assumption 7'.  $\underline{b} > -\infty$ .*

This is a case of practical interest, since applied work frequently restricts the space of allowable government savings.<sup>8</sup> Since such a restriction limits the ability of a government to carry out the parallel mimicking strategy discussed in section 2.2, leading our proof to break down, it is also natural to ask whether it could be a source of multiplicity.

In fact we show, by means of a simple example, that when  $\underline{b} = 0$  and  $V^d$  is the value of autarky, there may be multiple equilibria. This turns out to be a special case, however, since we are able to extend the uniqueness result whenever either  $\underline{b} < 0$  or  $V^d$  is strictly worse than the value of autarky.

**Potential multiplicity with no savings and autarky punishment** We first start from the observation that, when no government savings is allowed and default is punished by autarky, there is always an equilibrium without any debt.

**Lemma 8.** *When  $\underline{b} = 0$  and  $V^d(s) = V^{aut}(s) \equiv u(y(s), s) + \beta \mathbb{E}[V^{aut}(s') | s]$ , there exists an equilibrium where all default thresholds are identically equal to zero and the government never borrows:  $b^*(s) = 0$  for all  $s$  and  $b'(b, s) = 0$  for all  $b, s$ .*

The argument is straightforward: consider a government with some outstanding debt that cannot save or borrow again. This government can either default now, achieving the value of autarky, or repay its debt today and live off its endowment in the future, which is strictly worse. It therefore always chooses to default. Anticipating this behavior, creditors do not lend.

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<sup>7</sup>In parallel and independent work, [Bloise et al. \(2016\)](#) have established a sufficient condition under which the [Bulow and Rogoff \(1989\)](#) result survives in incomplete markets environments with a general asset market structure, and discuss examples where it fails (see also [Pesendorfer 1992](#)). This sufficient condition is a “high implied interest rates” condition, as in [Alvarez and Jermann \(2000\)](#). When the only available asset is a risk-free bond and the endowment process is bounded, this condition simplifies to  $R > 1$  (our assumption 6). Our result in this section therefore complements theirs, by exhibiting an explicit replication strategy with risk-free bonds, and reinterpreting the no-lending result as a result about equilibrium uniqueness.

<sup>8</sup>For example, [Chatterjee and Eyigungor \(2012\)](#) exclude savings from their grid, although they find in their numerical simulations that this restriction does not bind.

The no-debt equilibrium is not necessarily the only one, however, as the following proposition illustrates.

**Proposition 9.** *Suppose, as in lemma 8, that  $\underline{b} = 0$  and  $V^d(s) = V^{aut}(s)$ . Suppose also that  $\mathcal{S} = \{s_L, s_H\}$  with  $y(s_L) < y(s_H)$ ,  $\pi(s_L|s_H) = \pi(s_H|s_L) = 1$ ,  $u(c, s) = v(c)$  for strictly concave  $v$ , and  $R = 1/\beta$ . Then, for  $\beta$  sufficiently close to 1, there also exists an equilibrium where both default thresholds are strictly positive and the government sometimes borrows:  $b^*(s_L), b^*(s_H) > 0$  and  $b'(b, s) > 0$  for some  $b, s$ .*

Why does an equilibrium with debt exist? Suppose that the government can borrow at the risk-free rate  $R$ . Then, in this example, it would like to achieve constant consumption across periods, and can do so by borrowing in state  $s_L$  and repaying in state  $s_H$ . If  $\beta$  is close enough to one, the present value of smoothing endowment fluctuations in this way exceeds the one-off return from neglecting to repay, and the government chooses not to default. Anticipating repayment, creditors lend at rate  $R$  within the relevant range of  $b$ .

This multiplicity is a notable contrast to our uniqueness result.<sup>9</sup> Indeed, it embodies the intuition for multiplicity discussed in the introduction—an intuition that we rejected in section 2.2 for the  $\underline{b} = -\infty$  case. In the new example, expectations *can* be self-fulfilling. The equilibrium in lemma 8 has a vicious cycle where pessimistic creditors never lend and there is no incentive to repay, while the equilibrium in proposition 9 has a virtuous cycle where optimistic creditors lend on favorable terms and there is a strong incentive to repay.

We will now show, however, that this multiplicity is a special case. Moving away from the assumptions in lemma 8, either by allowing some savings or by adding a penalty for default, restores uniqueness.

**Proof of uniqueness** Before proceeding to a uniqueness result, we must make two additional assumptions on the environment.

*Assumption 8.* For each  $s$ ,  $u(c, s)$  is concave in  $c$ .

*Assumption 9.* There is some function  $v^d(s) \leq u(y(s), s)$  such that  $V^d(s) = v^d(s) + \beta \mathbb{E}[V^d(s') | s]$  for all  $s$ .

The concavity in assumption 8 is satisfied by most standard specifications of  $u$ , and it will be crucial to the modified proof strategy. Assumption 9 states that the value of default is the present discounted value of some flow utility  $v^d(s)$ , weakly less desirable than autarky.<sup>10</sup> This is common in the literature, which often assumes an output cost  $\tau(s)$  of default such that  $v^d(s) = u(y(s) - \tau(s), s)$ .

<sup>9</sup>Passadore and Xandri (2014) were the first to identify this type of multiplicity in a sovereign debt model without savings. Proposition 9 reframes this finding within the framework of this paper, and provides a simplified example.

<sup>10</sup>Imposing this structure involves some loss of generality, since we can no longer make the value of defaulting depend on state  $s$  without also affecting the value of being excluded from markets in state  $s$  after originally defaulting in state  $s' \neq s$ .



With these assumptions in hand, we can describe the new argument for uniqueness. This echoes the replication argument from section 2 in some respects, but there are also substantial differences. Rather than *mimicking at a distance*, which is no longer feasible, we use *compressed mimicking*. Given distinct revenue schedules  $Q$  and  $\tilde{Q}$  derived via (7) from default thresholds  $\{b^*(s)\}_{s \in \mathcal{S}}$  and  $\{\tilde{b}^*(s)\}_{s \in \mathcal{S}}$ , lemma 10 defines  $\lambda$  to be the minimum ratio between  $\tilde{b}^*(s) - \underline{b}$  and  $b^*(s) - \underline{b}$ . For any  $s$  and  $\tilde{b} - \underline{b} = \lambda(b - \underline{b})$ , a government starting at  $(\tilde{b}, s)$  can *compress* by  $\lambda$  the optimal strategy for a government (which we call the *target*) starting at  $(b, s)$ , choosing  $\tilde{b}(s^t) - \underline{b} = \lambda(b(s^t) - \underline{b})$  whenever the target government repays and defaulting whenever the target government defaults.

As in section 2, the mimicking government by construction obtains weakly better prices than the target government for its debt. Unlike in section 2, the mimicking government need not achieve higher consumption than the target government. Instead, because it is compressing the target's debt issuance plan by  $\lambda$ , in each period it obtains consumption  $\tilde{c}$  that is weakly higher than the convex combination  $\lambda c + (1 - \lambda)y(s)$  of the target's consumption  $c$  and state- $s$  autarky income  $y(s)$ . This inequality is strict when  $\underline{b} < 0$ , where the mimicking government can consume extra due to forgone financing costs. Concavity of  $u$  then implies that  $u(\tilde{c}, s)$  is strictly greater than  $\lambda u(c, s) + (1 - \lambda)u(y(s), s)$ . Summing the expected value across all periods, we obtain (17), the analog of (9); when  $\underline{b} = 0$ , the strict inequality can also follow from  $v^d(s) < u(y(s), s)$ .

**Lemma 10.** *Let  $Q$  and  $\tilde{Q}$  be two distinct revenue schedules, with  $Q$  reflecting expected default thresholds  $\{b^*(s)\}_{s \in \mathcal{S}}$  and  $\tilde{Q}$  reflecting expected default thresholds  $\{\tilde{b}^*(s)\}_{s \in \mathcal{S}}$ . Let  $V$  and  $\tilde{V}$  be the respective value functions for governments facing these revenue schedules. Define*

$$\lambda \equiv \min_s \frac{\tilde{b}^*(s) - \underline{b}}{b^*(s) - \underline{b}} \quad (16)$$

and assume, without loss of generality, that  $0 \leq \lambda < 1$ . Assume also either that  $\underline{b} < 0$ , or that  $v^d(s) < u(y(s), s)$  for all  $s$ . Then for any  $s$  and  $b$  such that  $V(b, s) \geq V^d(s)$ , we have

$$\tilde{V}(\tilde{b}, s) > (1 - \lambda)V^d(s) + \lambda V(b, s) \quad (17)$$

where  $\tilde{b} - \underline{b} \equiv \lambda(b - \underline{b})$ . This can equivalently be written as

$$\tilde{V}(\tilde{b}, s) - V^d(s) > \lambda \left( V(b, s) - V^d(s) \right) \quad (18)$$

In contrast to (9), inequality (17) in lemma 10 does not show that  $\tilde{V}(\tilde{b}, s)$  is higher than  $V(b, s)$ . Fortunately, this is not needed to establish uniqueness in proposition 11. Instead, (18) suffices to obtain a contradiction. Inequality (18) shows that if a government facing  $Q$  weakly prefers not to default at  $(b, s)$  (so that  $V(b, s) - V^d(s) \geq 0$ ), then a government facing  $\tilde{Q}$  must *strictly* prefer not to default at  $(\tilde{b}, s)$  (so that  $\tilde{V}(\tilde{b}, s) - V^d(s) > 0$ ). It is therefore impossible for both  $b$  and  $\tilde{b}$  to be default thresholds for their respective value functions.

**Proposition 11.** *If either  $\underline{b} < 0$ , or  $v^d(s) < u(y(s), s)$  for all  $s$ , Markov perfect equilibrium has a unique value function  $V(b, s)$  and bond revenue schedule  $Q(b, s)$ .*

*Proof.* If, to the contrary, we have distinct equilibria  $(V, Q)$  and  $(\tilde{V}, \tilde{Q})$  with default thresholds  $\{b^*(s)\}_{s \in \mathcal{S}}$  and  $\{\tilde{b}^*(s)\}_{s \in \mathcal{S}}$ , define  $\lambda$  as in (16) and assume without loss of generality that  $0 \leq \lambda < 1$ .

Let  $\bar{s}$  be the state where the minimum in (16) is obtained. Evaluating (18) at  $\tilde{b} = \tilde{b}^*(\bar{s})$ ,  $b = b^*(\bar{s})$ , and  $s = \bar{s}$ , we obtain

$$0 = \tilde{V}(\tilde{b}^*(\bar{s}), \bar{s}) - V^d(\bar{s}) > \lambda \left( V(b^*(\bar{s}), \bar{s}) - V^d(\bar{s}) \right) = 0$$

which is a contradiction. □

The crucial innovation of proposition 11 is that it departs slightly from the conditions in lemma 8, which turn out to be fragile. Rather than disallowing all savings and punishing default with autarky, the proposition assumes that either some saving is allowed or that the value from default is strictly worse than autarky. With either modification, the no-debt equilibrium in lemma 8 is eliminated, because the value of default is now strictly worse than the value of remaining in the market with zero debt, and the government will always opt to repay a sufficiently small debt. Essentially, the government needs *some* reason not to default, however small—and it may be either a carrot (access to saving) or a stick (losses from default). Either way, the no-debt equilibrium is eliminated, and at that point proposition 11 can establish uniqueness.<sup>11</sup>

## 4.2 Stochastic market reentry

In the literature, a very common departure from the benchmark model of Section 2 is an assumption that market reaccess is possible after default (for example Aguiar and Gopinath 2006 and Arellano 2008). This makes the value of default depend on the equilibrium value of borrowing, implying that lemma 2 and proposition 3 do not directly apply. Our argument can no longer completely establish uniqueness, but we are able to rule out the most commonly hypothesized form of multiplicity—the existence of distinct “favorable” and “adverse” equilibria, in which the favorable equilibrium offers uniformly better revenues  $Q$ . We also show uniqueness in the special case where states are independently and identically distributed.

To be concrete, suppose that it is possible to re-access markets with zero debt after a stochastic period of exclusion, which has independent probability  $1 - \lambda$  of ending in each period. That is, replace assumption 9 with<sup>12</sup>

<sup>11</sup>If we considered a case with a positive debt minimum ( $\underline{b} > 0$ ), the opposite logic would prevail: default would become more attractive because it would relieve the government of the requirement to service  $\underline{b}$ . This would make the existence of an always-default equilibrium more likely and raise the chances of multiplicity. However, since this benefit is difficult to interpret, we maintain our assumption 4 that  $\underline{b} \leq 0$ . See also footnote 3.

<sup>12</sup>This formulation is the one used by Arellano (2008) and Aguiar and Gopinath (2006). It does not encompass the possibility of recovery on defaulted debt or debt renegotiation (see for example Pitchford and Wright 2007, Yue 2010, or Arellano and Bai 2014). Multiple equilibria might result if multiple  $V^d$  are possible—for example as a result of

Assumption 9'.  $V^d$  satisfies

$$V^d(s) = v^d(s) + \beta\lambda\mathbb{E}_{s'} [V^d(s')] + \beta(1-\lambda)\mathbb{E}_{s'} [V^o(0, s')] \quad (19)$$

We also revert to assumption 7 ( $\underline{b} = -\infty$ ). In this framework, we can now prove the following specialized analog of proposition 3.

**Proposition 12.** *In the model with stochastic reentry, there do not exist two distinct equilibria  $(V, Q)$  and  $(\tilde{V}, \tilde{Q})$  such that  $Q(b, s) \geq \tilde{Q}(b, s)$  for all  $b$  and  $s$ .*

In general, the endogeneity of  $V^d(s)$  in (19) makes it difficult to analytically characterize equilibria. In the particular case examined by proposition 12, however, the proof strategy from proposition 3 still applies with some modification. The core insight is that if  $Q \geq \tilde{Q}$ , then  $V^d \geq \tilde{V}^d$ , because a government facing a uniformly better revenue schedule after reentry is better off. Furthermore, if  $Q$  and  $\tilde{Q}$  are distinct and  $Q \geq \tilde{Q}$ , there must be some  $s$  for which  $b^*(s) > \tilde{b}^*(s)$ . We then can apply the argument from lemma 2 and proposition 3, having a government in the  $(\tilde{V}, \tilde{Q})$  equilibrium mimic the strategy of a government in the  $(V, Q)$  equilibrium. The fact that  $\tilde{V}^d \leq V^d$  only helps our argument, since it is further reason why the government in the  $(\tilde{V}, \tilde{Q})$  equilibrium will prefer the mimicking strategy to default.

In short, when there is reentry, uniformly higher bond prices defeat themselves: they make default and eventual reentry more attractive, raising the probability of default and pushing bond prices back down.

Although we cannot prove uniqueness more generally, this result does rule out the popular hypothesis—as discussed in the introduction—that sovereign debt markets can vary between self-sustaining “favorable” and “adverse” equilibria. Instead, if multiplicity exists, we know that it must be a surprising kind of multiplicity: among any two equilibria, each must offer cheaper borrowing in some places and more expensive borrowing in others.

**Special case with iid exogenous state** It is possible to demonstrate full uniqueness in one special case. Suppose now that  $s$  follows an iid process with probability  $\pi(s)$ . It follows that the expected value from reentry  $\mathbb{E}_{s'}[V^o(0, s')]$  in (19) is independent of the states preceding  $s'$ , and we can denote this expectation by  $V^{re}$ . The iid assumption also implies that the bond revenue schedule  $Q$  depends only on the debt amount  $b'$ , not the current state  $s$ , as (7) reduces to

$$Q(b') = \frac{b'}{R} \sum_{\{s': b' \leq b^*(s')\}} \pi(s') \quad (20)$$

**Proposition 13.** *In the model with iid states and stochastic market reentry, Markov perfect equilibrium has a unique value function  $V(b, s)$  and bond revenue schedule  $Q(b)$ .*

Proposition 13 follows for reasons similar to proposition 12. For any distinct equilibria  $(V, Q)$  and  $(\tilde{V}, \tilde{Q})$ , the only difference between the default value functions  $V^d$  and  $\tilde{V}^d$  arises from the coordination on different bargaining equilibria, or more generally if  $V^d$  was endogenized as part of the game.

expected reentry value, which is now just a scalar  $V^{re}$ . Whichever equilibrium has the higher reentry value must have a more favorable bond revenue schedule, meaning that at least one of its default thresholds is higher. As with proposition 12, we can then invoke a mimicking argument to show that the equilibrium with a higher default value cannot also have a higher default threshold for some  $s$ .

This result further emphasizes how subtle any multiplicity in the model with reentry, if it exists, must be: it must rely, in some way, on the transition probabilities of the Markov process being non-iid.

### 4.3 Other variations on the model and multiplicity results

We have showed that the benchmark Eaton-Gersovitz model of sovereign debt with default does not admit multiple equilibria, and that this uniqueness result partly extends to the more complex environments of subsections 4.1 and 4.2. Nevertheless, multiplicity arises in several other sovereign debt models in the literature. This section reviews the ways in which these models sidestep the uniqueness result present in the benchmark framework.

Markov perfect equilibrium in the model we studied includes a revenue function  $Q(b', s)$ , which depends only on the current state  $s$  and the bond payment  $b'$  promised tomorrow. After observing  $s$ , in each period the government can choose either to default or to repay and sell some quantity  $b'$  of bonds for next period. Once the government chooses to repay and selects some  $b'$ , there is no uncertainty about the amount  $Q(b', s)$  that will be raised; no further choices are made until the next period, when the next state  $s'$  is realized and the process repeats itself. As presented in appendix A.2, this process can be explicitly written as a game between governments and risk-neutral investors. It is possible to define subgame perfect equilibria in this game, and proposition 4 shows that uniqueness still holds for these equilibria in the benchmark model.

Our uniqueness result can disappear if the timing and action space of the game are modified. For instance, in the model of Cole and Kehoe (2000), the government has the option to default after observing the outcome of the current period's bond auction. If it defaults, it can keep the proceeds of the auction but avoid repayment on its maturing debt. Given enough risk aversion, this option is preferable when the current period's auction yields little revenue, and the cost of repaying maturing debt out of current-period resources is prohibitively high. A coordination problem among creditors thus emerges, leading to multiple equilibria: they might either offer high prices, in which case the government will repay, or offer low prices, in which case the government will default and thereby justify the low prices. The literature sometimes refers to this phenomenon as "rollover multiplicity". It is absent in the model we study, which excludes the option to default after revenue from the auction comes in; but it captures an important intuition, which is that rolling over large amounts of short-term debt can be a source of fragility.<sup>13</sup>

<sup>13</sup>Our results explain why the emerging quantitative literature evaluating the importance of non-fundamental forces in explaining the recent Eurozone crisis (Conesa and Kehoe 2015, Bocola and Dovis 2015, Ayres, Navarro, Nicolini and Teles 2015) has turned to a Cole-Kehoe formulation of the timing: even in more complex quantitative models, the

In the model of [Calvo \(1988\)](#), multiplicity arises because of the way the bond revenue-raising process works. In the Calvo model, a government borrows an exogenous amount  $b$  at date 0 and inherits a liability of  $R_b b$  at date 1. It then uses a mix of distortionary taxation and debt repudiation to finance a given level of government spending. Since a higher interest rate  $R_b$  tilts the balance towards more repudiation at date 1, and since investors need to break even when lending to the government, there exist two rational expectations equilibria: one with high  $R_b$  and high repudiation, and one with low  $R_b$  and low repudiation. This is sometimes called “Laffer curve multiplicity” in reference to the shape of the bond revenue curve that arises in this model (the function that gives bond revenue  $b$  as a function of promised repayment  $R_b b$  has an inverted-V shape). In the model we study, the government directly announces the amount it will owe tomorrow, allowing it to avoid the downward-sloping part of the bond revenue curve.<sup>14</sup> [Lorenzoni and Werning \(2014\)](#) make a forceful argument that such an assumption requires a form of commitment that governments are unlikely to have: in practice, if they raise less auction revenue than expected, they may auction additional debt rather than making the burdensome fiscal adjustments that are otherwise necessary.

In effect, both the rollover multiplicity of [Cole and Kehoe \(2000\)](#) and the Laffer curve multiplicity of [Calvo \(1988\)](#) emerge from a more elaborate game between governments and investors. They create self-fulfilling alternate equilibria by allowing governments to act in ways ruled out by the game-theoretic formulation of the benchmark Eaton-Gersovitz model: when auction revenue is insufficient, governments can either take the revenue and then default (as in [Cole and Kehoe 2000](#)) or dilute investors by issuing more debt in the same period (as in Lorenzoni and Werning’s interpretation of [Calvo 1988](#)). Since the Eaton-Gersovitz model alone cannot produce multiplicity, these modifications to the game may prove important to interpreting any multiplicity we see in practice. More generally, they suggest that a detailed look at institutions, and the practical options available to sovereign debtors when they raise funds in debt markets, is necessary to understand when the Eaton-Gersovitz model succeeds and when it fails as a benchmark.

Another important strand of the literature considers long-term debt, as in [Hatchondo and Martinez \(2009\)](#). Here, our mimicking-based proof of uniqueness breaks down, since bond prices are influenced by the likelihood of endogenous default in the arbitrarily distant future. Indeed, in recent work, [Aguiar and Amador \(2016\)](#) have constructed an example of multiplicity with long-term debt. This provides a contrast to our result—and in a surprising direction, since informal discussions often suggest that multiplicity is more likely, not less, with short-term debt.

A final route to multiplicity is the possibility of dynamic inefficiency. When  $R \leq 1$  and assumption 6 is violated, classical results on the possibility of bubbles in dynamically inefficient economies lead us to expect the possibility of multiple equilibria. Indeed, in a model with complete markets, [Hellwig and Lorenzoni \(2009\)](#) exhibit an equilibrium with  $R = 1$  and self-sustaining

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Eaton-Gersovitz timing tends to generate uniqueness and is therefore poorly suited for such an exercise.

<sup>14</sup>Interestingly, the setup of the original [Eaton and Gersovitz \(1981\)](#) model does not let the government choose on the bond revenue curve *a priori*, although their analysis focuses on equilibria in which it effectively does.

debt in spite of a Bulow-Rogoff punishment for exclusion.<sup>15</sup>

## 5 Conclusion

We prove that the Eaton-Gersovitz model and several of its variants have a unique equilibrium. Our results settle an important outstanding question in the literature, making use of a replication-based proof that may be applicable more generally. By showing that no changes in a government's reputation for repayment can be self-sustaining, we rule out a widely suspected source of multiple equilibria in sovereign debt markets. We hope that future research will build on this result, exploring the extent to which alternative economic mechanisms—for instance, long-term debt, risk-averse lenders, a richer supply side, or partial recovery by creditors—might either reinforce uniqueness or generate multiplicity.

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<sup>15</sup>Note that  $R = 1$  arises endogenously in their general equilibrium setting (see also [Jeske \(2006\)](#) and [Wright \(2006\)](#)).

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## A Proofs

### A.1 Proof of proposition 1 (existence of Markov perfect equilibrium)

We prove existence of Markov perfect equilibrium constructively, following a fixed point procedure similar to the one typically used by the sovereign debt literature to find an equilibrium. Section A.1.1 defines a functional  $\mathbb{B}(V)$  mapping value functions to default thresholds, and proves properties of this mapping. Section A.1.2 defines a functional  $\mathbb{V}(B)$  mapping default thresholds to value functions, and proves properties of that mapping. Finally, Section A.1.2 shows that iterating on the operator  $\mathbb{T} = \mathbb{B} \circ \mathbb{V}$ , starting from thresholds identically equal to zero, produces a limit set of default thresholds that constitute an equilibrium.

#### A.1.1 Default thresholds for given $V: \mathbb{B}(V)$

Consider a set of  $S$  strictly decreasing, continuous functions  $V(b, s)$ . For each state  $s$ , define the threshold  $b^*(s)$  as  $-\infty$  if  $\sup_b V(b, s) < V^d(s)$ , or  $+\infty$  if  $\inf_b V(b, s) > V^d(s)$ . In other cases, let  $b^*(s)$  be equal to the unique solution to

$$V(b^*(s), s) = V^d(s)$$

This defines a functional  $\mathbb{B}(V)$ . The following shows that this is a monotone mapping, and provides conditions on  $V$  under which  $\mathbb{B}(V)$  is positive and bounded.

**Lemma 14.** *The following propositions hold for every  $s$ .*

- a) If  $V(0, s) \geq V^{nb}(0, s)$ , then  $b^*(s) \geq 0$
- b) If  $V(b, s) = -\infty$ , then  $b^*(s) < b$
- c) If  $V^A(b, s) \geq V^B(b, s)$  for all  $b$ , then the respective default thresholds satisfy  $b^{*A}(s) \geq b^{*B}(s)$

*Proof.* The proof follows because  $V$  is continuous and strictly decreasing. Assumption 5 guarantees that  $V(0, s) \geq V^{nb}(0, s) \geq V^d(s) = V(b^*(s), s)$ , so **a)** holds. Assumption 5 also guarantees that  $V^d(s)$  is finite, so  $V(b, s) < V(b^*(s), s)$ , and **b)** holds. Finally,  $V^B(b^{*B}(s), s) = V^d(s) = V^A(b^{*A}(s), s) \geq V^B(b^{*A}(s), s)$ , so **c)** holds.  $\square$

### A.1.2 Value functions $V$ given default thresholds: $\mathbb{V}(B)$

Consider now a set of positive default thresholds  $B = \{b^*(s)\}$ ,  $b^*(s) \geq 0$ . Define  $V$  as the solution to

$$\begin{aligned}
 V(b, s; \{b^*(s')\}) &\equiv \sup_{b(s^t), p(s^t) \in \{0,1\}} \left\{ \sum_{s^t} \beta^t \Pi(s^t) u(c(s^t), s_t) \mathbf{1}_{\{p(s^t)=1\}} \right. \\
 &\quad \left. + \sum_{s^t} \beta^t \Pi(s^t) V^d(s_t) \mathbf{1}_{\{p(s^t)=0, p(s^{t-1})=1\}} \right\} \tag{21} \\
 \text{s.t. } c(s^t) &= y(s_t) + \frac{b(s^t)}{R} \left( \sum_{s_{t+1}} \pi(s_{t+1}|s_t) \cdot \mathbf{1}_{\{b(s^t) \leq b^*(s_{t+1})\}} \right) - b(s^{t-1}) \\
 p(s^t) &\leq p(s^{t-1}) \\
 p(s_0) &= 1 \\
 b(s^{-1}) &= b \\
 s_0 &= s
 \end{aligned}$$

This defines a mapping  $\mathbb{V}(B)$  from default thresholds to value functions. We now prove properties of this mapping, including monotonicity and continuity.

**Lemma 15.** *The following propositions hold for every  $s$ .*

- a) The supremum in (21) is attained for any  $b$ , and  $V(b, s) < \infty$
- b)  $V(b, s)$  is strictly decreasing in  $b$
- c)  $V(b, s)$  is continuous in  $b$  for every  $s$
- d)  $V(b, s; \{b^*(s')\})$  is increasing in  $\{b^*(s')\}$
- e)  $V(0, s) \geq V^{nb}(0, s)$
- f)  $V\left(\frac{\bar{b}}{R} + y(s), s\right) = -\infty$
- g)  $V(b, s; \{b^*(s')\})$  is continuous in  $\{b^*(s')\}$

*Proof.* We prove each of the propositions in turn.

- a) We restrict ourselves to cases where such that  $V(b, s) > -\infty$ , otherwise the proposition is trivial. We prove that the maximum is attained by showing that the problem in (21) is the maximization of an upper semicontinuous function on a compact set, and exhibit an upper bound to show  $V(b, s) < \infty$ . First, assumption 3 guarantees that  $c(s^t) > 0$ , which (given that assets receive the risk-free rate) bounds the rate of growth of assets: there exists  $D > 0$  such that  $b(s^t) \geq -DR^{t+1}$ . Together with assumption 4, this guarantees that  $b(s^t)$  must be chosen on a compact interval  $[-DR^{t+1}, \bar{b}]$ , and hence that the set of all arguments

$\{b(s^t), p(s^t)\}$  is compact. Second, these bounds on  $b(s^t)$  place an upper bound on  $c(s^t)$  which, together with assumption 2, yields a bound on flow utility,  $\beta^t u(c(s^t), s_t) \leq (\beta R^\kappa)^t \bar{u}$  where  $\bar{u} < \infty$ . Third, the presence of the default option implies that flow utility along the no-default path is bounded below in all periods,  $\beta^t u(c(s^t), s_t) \geq \beta^t \underline{u}$  for  $\underline{u} > -\infty$ . Summing up, we have bounds on flow utility:

$$\beta^t \underline{u} \leq \beta^t u(c(s^t), s_t) \leq (\beta R^\kappa)^t \bar{u} \quad (22)$$

Next, all partial sums in the maximand (21) are upper semicontinuous in the argument. This follows from the fact that they consist entirely of continuous functions except  $\mathbf{1}_{\{b(s^t) \leq b^*(s_{t+1})\}}$ , which is upper semicontinuous. Inequality (22) together with  $\beta < 1$  and  $\beta R^\kappa < 1$  allows one to apply the Weierstrass M-test to conclude that the sum converges uniformly, and hence that the limit is also upper semicontinuous in the argument. Hence the maximum in (21) is attained. Finally, (22) together with the fact that default values are finite guarantee that the objective in (21) is uniformly bounded from above, and hence the maximum  $V(b, s) < \infty$  as well.

- b) Fix  $s$  and consider  $\tilde{b} > b$ . Consider the optimal plan  $\{\tilde{b}(s^t), \tilde{p}(s^t)\}$  starting at  $(\tilde{b}, s)$ . Then the plan  $\{\tilde{b}(s^t), \tilde{p}(s^t)\}$  is also feasible starting at  $(b, s)$ , so that, letting  $Q = \frac{\tilde{b}(s^0)}{R} \sum_{\{s': \tilde{b}(s^0) \leq b^*(s')\}} \pi(s'|s)$ , we have

$$\begin{aligned} V(b, s) - V(\tilde{b}, s) &\geq u(y(s) + Q - b, s) - u(y(s) + Q - \tilde{b}, s) \\ &> u(y(s) + Q - \tilde{b}, s) - u(y(s) + Q - \tilde{b}, s) = 0 \end{aligned}$$

- c) Fix  $(b, s)$  and let  $\epsilon > 0$ . We show that (i) there exists  $\delta_1 > 0$  such that for any  $b < \tilde{b} < b + \delta_1$ ,  $V(\tilde{b}, s) > V(b, s) - \epsilon$ , and (ii) there exists  $\delta_2 > 0$  such that for any  $b > \tilde{b} > b - \delta_2$ ,  $V(\tilde{b}, s) < V(b, s) + \epsilon$ . Together with  $V$  being strictly decreasing, (i) and (ii) establish continuity. For (i), consider the optimal plan  $\{b(s^t), p(s^t)\}$  starting at  $(b, s)$ . This plan is also feasible starting at  $(\tilde{b}, s)$  and delivers the same consumption at every point except  $t = 0$ , where consumption is  $\tilde{b} - b$  lower. Hence letting  $c(s^0)$  be the  $t = 0$  consumption level for the optimal plan starting at  $(b, s)$ , we know

$$V(b, s) - V(\tilde{b}, s) = u(c(s^0)) - u(c(s^0) - \delta_1)$$

will be  $< \epsilon$  as desired if  $\delta_1 > 0$  is defined via continuity of  $u$  such that  $|u(c) - u(c(s^0))| < \epsilon$  for all  $|c - c(s^0)| < \delta_1$ .

For (ii), we must appeal to a uniform continuity argument to choose  $\delta_2$ . We first find a compact set  $[\underline{c}, \bar{c}]$  such that any optimal plan with  $\tilde{b} < b$  (and hence  $V(\tilde{b}, s) > V(b, s)$ ) has first period consumption  $\tilde{c}(s^0) \in [\underline{c}, \bar{c}]$ . To do this, recall from A.1.2 that the sum of all terms in (21) for  $t \geq 1$  is bounded from above by an upper bound  $\bar{V} < \infty$ . Hence the initial

consumption level  $\tilde{c}(s^0)$  associated with an optimum  $V(\tilde{b}, s) > V(b, s)$  must be such that

$$u(\tilde{c}(s^0), s_0) + \bar{V} \geq V(b, s) \quad (23)$$

From assumption 3,  $u(\tilde{c}(s^0), s_0) \rightarrow -\infty$  as  $\tilde{c}(s^0) \rightarrow 0$ , and hence for (23) to be satisfied we must have  $\tilde{c}(s^0) \geq \underline{c} > 0$  for some lower bound  $\underline{c}$ . We also know that  $\tilde{c}(s^0) \leq \bar{b}/R + \bar{y} - b(s^0) \equiv \bar{c}$ , giving us an upper bound. Since  $u$  is continuous and  $[\underline{c}, \bar{c}]$  is a compact interval, we can pick a single  $\delta_2 > 0$  such that  $|u(c_A, s_0) - u(c_B, s_0)| < \epsilon$  for all  $c_A \in [\underline{c}, \bar{c}]$  and  $|c_B - c_A| < \delta_2$ .

Now consider the optimal plan  $\{\tilde{b}(s^t), \tilde{p}(s^t)\}$  starting at  $(\tilde{b}, s)$ . This plan is also feasible starting at  $(b, s)$  and delivers the same consumption at every point except  $t = 0$ , where consumption is  $b - \tilde{b}$  lower. Hence we have

$$V(\tilde{b}, s) - V(b, s) = u(\tilde{c}(s^0), s_0) - u(\tilde{c}(s^0) - \delta_2, s_0) < \epsilon$$

as desired.

- d) Since  $b^*(s') \geq 0$ , increasing  $b^*(s')$  always weakly increases  $\frac{b(s^t)}{R} \left( \sum_{s_{t+1}} \pi(s_{t+1}|s_t) \cdot \mathbf{1}_{\{b(s^t) \leq b^*(s_{t+1})\}} \right)$  when  $b(s^t) \geq 0$  and leaves it unchanged when  $b(s^t) \leq 0$ , which completes the proof.
- e) Follows from d), since  $V^{nb}(0, s)$  is the value with default thresholds all equal to zero, as shown in section 3.2.
- f) Assumption 4 ensures that for any  $b' > 0$ ,  $\frac{b'}{R} \sum_{\{s': b' \leq b^*(s')\}} \pi(s'|s) < \frac{\bar{b}}{R}$ . Hence feasible consumption at date 0 is  $c(s^0) < y(s) + \frac{\bar{b}}{R} - \left( \frac{\bar{b}}{R} + y(s) \right) = 0$ . Given that the continuation value for any  $b'$  is finite, f) follows from assumption 3.
- g) Fix  $b$  and  $s^0$ . Let  $\epsilon > 0$  and let  $\{b^*(s')\}$  be a set of default thresholds. In an argument similar to the proof of c), we show that (i) there exists  $\delta_1$  such that, for any alternative set of default thresholds  $\{\tilde{b}^*(s')\}$  such that  $|b^*(s') - \tilde{b}^*(s')| < \delta_1$  for all  $s'$ , we have  $V(b, s, \{\tilde{b}^*(s')\}) > V(b, s, \{b^*(s')\}) - \epsilon$ , and (ii) there exists  $\delta_2$  such that, for any  $\{\tilde{b}^*(s')\}$  such that  $|b^*(s') - \tilde{b}^*(s')| < \delta_2$  for all  $s'$ , we have  $V(b, s, \{b^*(s')\}) > V(b, s, \{\tilde{b}^*(s')\}) - \epsilon$ . Combining (i) and (ii) then proves continuity. In both cases, we use the fact that a government facing debt thresholds that are lower by at most  $\delta$  can guarantee itself a consumption plan that is only  $\delta$  below that of a government with reference debt thresholds at date 0—and above at every other date—using a mimicking strategy, as embodied in the following claim.

*Claim.* Assume that  $|b^*(s') - \tilde{b}^*(s')| < \delta$ . Let  $\{b(s^t), p(s^t)\}$  be a plan that achieves consumption  $c(s^t)$  subject to the default thresholds  $\{b^*(s')\}$  starting from  $(b, s^0)$ . Then there is another plan  $\{\tilde{b}(s^t), p(s^t)\}$  that achieves consumption  $\tilde{c}(s^t)$  subject to the default thresholds  $\{\tilde{b}^*(s')\}$  such that  $\tilde{c}(s^t) > c(s^t)$  for all  $t \geq 1$  and  $\tilde{c}(s^0) > c(s^0) - \delta$ .

*Proof of claim.* Define  $\tilde{b}(s^t) \equiv b(s^t) - \delta$  for all  $t \geq 0$  and  $\tilde{b}(s^{-1}) \equiv b(s^{-1}) = b$ . Then compute

$$\begin{aligned}\tilde{c}(s^t) &= y(s_t) + \frac{b(s^t) - \delta}{R} \left( \sum_{s_{t+1}} \pi(s_{t+1}|s_t) \cdot \mathbf{1}_{\{b(s^t) - M \leq \tilde{b}^*(s_{t+1})\}} \right) - (b(s^{t-1}) - \delta) \\ &\geq y(s_t) + \frac{b(s^t)}{R} \left( \sum_{s_{t+1}} \pi(s_{t+1}|s_t) \cdot \mathbf{1}_{\{b(s^t) \leq b^*(s_{t+1})\}} \right) - b(s^{t-1}) + \left(1 - \frac{1}{R}\right) \delta > c(s^t)\end{aligned}$$

and

$$\begin{aligned}\tilde{c}(s^0) &= y(s_0) + \frac{b(s^0) - \delta}{R} \left( \sum_{s_1} \pi(s_1|s_0) \cdot \mathbf{1}_{\{b(s^0) - M \leq \tilde{b}^*(s_1)\}} \right) - b(s^{-1}) \\ &\geq y(s_0) + b(s^0) \left( \sum_{s_1} \pi(s_1|s_0) \cdot \mathbf{1}_{\{b(s^0) \leq b^*(s_1)\}} \right) - b(s^{-1}) - \frac{\delta}{R} > c(s^0) - \delta\end{aligned}$$

□

To prove (i), consider the plan  $\{c(s^t), b(s^t), p(s^t)\}$  that achieves  $V(b, s, \{b^*(s')\})$ . Using the continuity of  $u(c, s^0)$ , let  $\delta_1$  be such that  $|u(c', s_0) - u(c(s^0), s_0)| < \epsilon$  for all  $|c' - c(s^0)| < \delta_1$ . Then, whenever the thresholds  $\{\tilde{b}^*(s')\}$  are such that  $|b^*(s') - \tilde{b}^*(s')| < \delta_1$  for all  $s'$ , it follows from the claim that there is a consumption plan  $\{\tilde{b}(s^t), p(s^t)\}$  for these thresholds that achieves consumption above  $c(s^0) - \delta$  in the first period and above  $c(s^t)$  everywhere else, and hence value greater than  $V(b, s, \{b^*(s')\}) - \epsilon$ .

To prove (ii), suppose for some  $\{\tilde{b}^*(s')\}$  that  $V(b, s, \{\tilde{b}^*(s')\}) \geq V(b, s, \{b^*(s')\})$  (otherwise, the desired inequality is immediate), and let  $\{\tilde{b}(s^t), p(s^t)\}$  be the plan attaining the optimum for  $V(b, s, \{\tilde{b}^*(s')\})$ . We can establish using the argument from the proof of **c)** we can pick a single  $\delta_2 > 0$  such that  $|u(c, s_0) - u(\tilde{c}(s_0), s_0)| < \epsilon$  for  $|c - \tilde{c}(s_0)| < \delta_2$ . It follows from the claim that there is a plan  $\{\tilde{\tilde{b}}(s^t), p(s^t)\}$  that (subject to the default thresholds  $\{b^*(s')\}$ ) achieves consumption  $\tilde{\tilde{c}}(s^t)$  that is strictly greater than  $\tilde{c}(s^t)$  for all  $t \geq 1$  and strictly greater than  $\tilde{c}(s^0) - \delta_2$  for  $t = 0$ . From the choice of  $\delta_2$  we know that  $|u(\tilde{\tilde{c}}(s^0), s_0) - u(\tilde{c}(s^0), s_0)| < \epsilon$ , and hence that the proposed plan  $\{\tilde{\tilde{b}}(s^t), p(s^t)\}$  gives value strictly greater than  $V(b, s, \{\tilde{b}^*(s')\}) - \epsilon$ . It follows that  $V(b, s, \{b^*(s')\}) > V(b, s, \{\tilde{b}^*(s')\}) - \epsilon$  as desired.

□

### A.1.3 Existence of equilibrium

Using the operators defined in Sections **A.1.1** and **A.1.2**, we can define the operator  $\mathbb{T} = \mathbb{B} \circ \mathbb{V}$ .

**Lemma 16.** *The operator  $\mathbb{T}$  is monotone increasing and maps the set  $\prod_s \left[0, y(s) + \frac{\bar{b}}{R}\right]$  onto itself*

*Proof.* Monotonicity follows by combining lemmas **14c)** and **15d)**. By combining lemmas **14a)** and **15e)**, we obtain that  $\mathbb{T}b^*(s) \geq 0$  whenever  $b^*(s) \geq 0$ . By combining lemmas **14b)** and **15f)**, we obtain that  $\mathbb{T}b^*(s) \leq y(s) + \frac{\bar{b}}{R}$  for each  $s$ . □

Let  $b^{*0}(s) = 0$  for every  $s$ . For  $n \geq 1$  define the sequence

$$b^{*n} = \mathbb{T}b^{*(n-1)}$$

By lemma 16, the sequences  $b^{*n}(s)$  are increasing and bounded for every  $s$ . Hence they converge to form a set of thresholds  $\{b^{*\infty}\}$ . Define  $V^n = \mathbb{V}(b^{*n})$  and  $V^\infty = \mathbb{V}(b^{*\infty})$ . From Lemma 15g) it follows that  $V^\infty(b, s) = \lim_{n \rightarrow \infty} V^n(b, s)$ . Next, because  $V^n$  is a sequence of continuous bijective functions with continuous inverses, whose limit  $V^\infty$  is continuous and bijective, and since by definition  $\mathbb{B}(V^n)(s) = (V^n)^{-1}(V^d(s), s)$ , we have that

$$\mathbb{B}\left(\lim_{n \rightarrow \infty} V^n\right) = \lim_{n \rightarrow \infty} \mathbb{B}(V^n)$$

and therefore

$$\mathbb{B}(V^\infty) = \lim_{n \rightarrow \infty} \mathbb{T}b^{*n} = b^{*\infty} \quad (24)$$

So  $(V^\infty, b^{*\infty})$  constitutes an equilibrium, as we set out to prove. To map these objects to those in the main text, define  $V = V^\infty$  and the bond revenue schedule  $Q$  as

$$Q(b', s) = \frac{b'}{R} \mathbb{P}_{s'|s} [b' \leq b^{*\infty}(s')] = \frac{b'}{R} \sum_{\{s': b' \leq b^{*\infty}(s')\}} \pi(s'|s)$$

then  $(V, Q)$  is a Markov perfect equilibrium, since (2)-(3) is the recursive formulation of the problem in (21) for the schedule  $Q$  generated by the thresholds  $\mathbb{B}(V)$ , and (24) guarantees that (4) holds.

## A.2 Proof of proposition 4 (uniqueness of subgame perfect equilibrium)

This appendix proves uniqueness of the subgame perfect equilibrium in the game of section 2. In order to define the game explicitly, we assume that there exist overlapping generations of two-period lived international investors. The set of investors born at time  $t$  is denoted by  $\mathcal{I}_t$ . We assume that  $\mathcal{I}_t$  is finite, that  $|\mathcal{I}_t| \geq 2$ , and that all investors are risk-neutral with preferences given by

$$-q_t a_{t+1}^i + \frac{1}{R} \mathbb{E}_t [a_{t+1}^i p_{t+1}] \quad (25)$$

where  $R > 1$ . We next describe the sequence of actions.

Every period, with incoming history  $h^{t-1}$ , after Nature realizes the exogenous state  $s_t$ , the government chooses repayment  $p_t$ . If it chooses  $p_t = 0$  (default), it obtains value  $V^d(s_t)$ , investors receive zero, and the game ends.

If it chooses  $p_t = 1$ , the government receives income  $y(s_t) \geq 0$  and chooses next period debt  $b_{t+1}$ . Next, every investor  $i$  simultaneously bids a price  $q_t^i \geq 0$  for the government's debt. Given



bids  $q_t^i$ , an auctioneer allocates the bonds  $a_{t+1}^i$  according to the following rule:

$$a_{t+1}^i = \begin{cases} \frac{b_{t+1}}{J} & \text{if } q_t^i = \max_{i'} q_t^{i'} \\ 0 & \text{otherwise} \end{cases}$$

where  $J$  is the number of investors bidding the maximum price. History for period  $t$  is now  $h^t = (h^{t-1}, s_t, b_{t+1}, \{q_t^i\})$ .

The government receives  $Q_t = q_t b_{t+1}$  where  $q_t = \max_{i'} q_t^{i'}$  and repays debt  $b_t$  to previous investors. Its consumption is then

$$c_t = y(s_t) - b_t + q_t b_{t+1}$$

for which it receives flow utility  $u(c_t, s_t)$ , and expected value

$$V(h^{t-1}, s_t) = \begin{cases} u(c_t, s_t) + \beta \mathbb{E}_t [V(h^t, s_{t+1})] & \text{if } p_t = 1 \\ V^d(s_t) & \text{if } p_t = 0 \end{cases} \quad (26)$$

*Definition 2.* A government strategy is  $p(h^{t-1}, s_t)$ ,  $b'(h^{t-1}, s_t)$  specifying the repayment and next period debt decision after each history  $h^{t-1}$  and state  $s_t$ . A strategy of investor  $i$  born at time  $t$  is a price bid  $q^i(h^{t-1}, s_t, b_{t+1})$ .

Together, investor strategies imply a bond revenue function  $Q(h^{t-1}, s_t, b_{t+1})$ .

*Definition 3.* A subgame perfect equilibrium consists of strategies for the government and investors such that at each  $(h^{t-1}, s_t)$ :

- a)  $p(h^{t-1}, s_t)$ ,  $b'(h^{t-1}, s_t)$  maximize (26)
- b) For all  $i \in \mathcal{I}_t$ ,  $q^i(h^{t-1}, s_t, b_{t+1})$  maximizes (25)

In any subgame perfect equilibrium, investor maximization leads to

$$q(h^{t-1}, s_t, b_{t+1}) = \frac{1}{R} \mathbb{E}_t [p(h^t, s_{t+1})] \quad (27)$$

We retain the other assumptions from the model in section 2 on  $u$ ,  $V^d$ , and the no-Ponzi bound on debt  $\bar{b}$ . These include assumption 1 and assumptions 1 through 5. Importantly, assumption 5 continues to imply that a government with debt  $b < 0$  never finds it optimal to default, so  $q(h, s, b') = \frac{1}{R}$  for any  $b' < 0$ .

The following lemma is crucial to the proof of unique equilibrium. It shows that in equilibrium, regardless of the history of play, a government with a strictly lower level of debt can always achieve a weakly higher value than a government with more debt in the same state, and is also weakly more likely to repay. Like the proof of lemma 2, it uses a mimicking-based argument, although here the proof is written in a recursive setting and must deal with technical complications that arise from the more general notion of equilibrium.

**Lemma 17.** Consider two subgame perfect equilibria  $A$  and  $B$ . For any  $(h_A, h_B, s)$ , if  $b(h_A) > b(h_B)$  then  $V_A(h_A, s) \leq V_B(h_B, s)$ , and  $p_B(h_B, s) = 1$  if  $p_A(h_A, s) = 1$ .

*Proof.* Define

$$M \equiv \sup_{h_A, h_B, s} \{b(h_A) - b(h_B) \quad \text{s.t.} \quad V_A(h_A, s) \geq V_B(h_B, s) \quad \text{and} \quad p_A(h_A, s) = 1\}$$

Assume  $M > 0$ .<sup>16</sup> Let  $0 < \epsilon < \frac{R-1}{R+1}M$ , and let  $(h_A, h_B, s)$  be such that  $V_A(h_A, s) \geq V_B(h_B, s)$ ,  $p_A(h_A, s) = 1$  and  $b(h_A) > b(h_B) + M - \epsilon$ . Define

$$\tilde{b}'_B = b'_A(h_A, s) - M - \epsilon \tag{28}$$

and continuation histories

$$\begin{aligned} h'_A &= \left( h_A, s, b'_A(h_A, s), \{q_A^i\} \right) \\ \tilde{h}'_B &= \left( h_B, s, \tilde{b}'_B, \{\tilde{q}_B^i\} \right) \end{aligned}$$

This is a feasible choice for the  $B$  government at  $(h_B, s)$  because we assume that debt can be chosen at any level below some upper bound. We aim to prove that through this choice of  $\tilde{b}'_B$ , the government in the  $B$  equilibrium achieves expected utility strictly greater than  $V_A(h_A, s)$ , thus establishing that  $V_B(h_B, s) > V_A(h_A, s)$ , a contradiction. We first establish that continuation utility for  $B$  is weakly greater in each future state, and then that current consumption is strictly greater, than their corresponding values for  $A$ .

We have, for all  $s' \in \mathcal{S}$ ,

$$V_B(\tilde{h}'_B, s') \geq V_A(h'_A, s') \tag{29}$$

Indeed, if  $p_A(h'_A, s') = 0$ , then immediately  $V_B(\tilde{h}'_B, s') \geq V^d(s') = V_A(h'_A, s')$ . Moreover, if  $p_A(h'_A, s') = 1$  then, since  $b(h'_A) - b(\tilde{h}'_B) > M$  by (28), we must have  $V_B(\tilde{h}'_B, s') > V_A(h'_A, s') \geq V^d(s')$ .

This last observation also implies that  $p_B(\tilde{h}'_B, s') = 1$  whenever  $p_A(h'_A, s') = 1$ . Hence, using the pricing condition (27), we also have

$$q_B(h_B, s, \tilde{b}'_B) \geq q_A(h_A, s, b'_A) \tag{30}$$

Using (30), we now show that the consumption achieved by  $B$  from the choice of  $\tilde{b}'_B$  is strictly greater than that achieved by  $A$ . Indeed, using the flow budget constraints of both governments,

<sup>16</sup>One can rule out the case  $M = \infty$  through a more direct mimicking argument: whenever  $b(h_A) - b(h_B) > \bar{b}$ , where  $\bar{b}$  is the upper bound on debt, then a government at  $(h_B, s)$  can mimic at distance  $\bar{b}$  the strategy of a government at  $(h_A, s)$ , with weakly more favorable prices (and hence strictly higher consumption due to its lower  $b$ ) guaranteed because it will never be in debt.

and dropping dependence on history for ease of notation:

$$\begin{aligned}\tilde{c}_B &= c_A + b_A - b_B + \tilde{q}_B \tilde{b}'_B - q_A b'_A \\ &\geq c_A + M - \epsilon + (\tilde{q}_B - q_A) b'_A + \tilde{q}_B (\tilde{b}'_B - b'_A)\end{aligned}\quad (31)$$

where the inequality follows from the definition of  $A$  and  $B$ .

Now if  $b'_A < 0$  then, since  $\tilde{b}'_B \leq b'_A < 0$  as well we have  $q_A = \tilde{q}_B = \frac{1}{R}$ , and hence  $(\tilde{q}_B - q_A) b'_A = 0$ . If  $b'_A \geq 0$  then using (30),  $(\tilde{q}_B - q_A) b'_A \geq 0$ .

Moreover, from (28),  $\tilde{b}'_B - b'_A = -M - \epsilon$ , and using  $\tilde{q}_B \leq \frac{1}{R}$ ,  $\tilde{q}_B (\tilde{b}'_B - b'_A) \geq -\frac{1}{R} (M + \epsilon)$ . Using these inequalities in (31),

$$\begin{aligned}\tilde{c}_B &\geq c_A + M - \epsilon - \frac{1}{R} (M + \epsilon) \\ &\geq c_A + \left(1 - \frac{1}{R}\right) M - \epsilon \left(1 + \frac{1}{R}\right) \\ &> c_A\end{aligned}\quad (32)$$

where the last line follows from the choice of  $\epsilon$ .

Since the utility from choosing  $\tilde{b}'_B$  provides a lower bound on  $V_B(h_B, s)$ , we have

$$\begin{aligned}V_B(h_B, s) &\geq u(\tilde{c}_B, s) + \beta \sum_{s'} V_B(\tilde{h}'_B, s') \\ &> u(c_A, s) + \beta \sum_{s'} V_A(h'_A, s') = V_A(h_A, s)\end{aligned}$$

where the second line follows from (29) and (32). This contradicts  $M > 0$ . Hence  $M \leq 0$ . We have proved that for  $(h_A, h_B, s)$ , if  $V_A(h_A, s) \geq V_B(h_B, s)$  and  $p_A(h_A, s) = 1$  then  $b(h_A) \leq b(h_B)$ .

So if  $b(h_A) > b(h_B)$ , either  $p_A(h_A, s) = 0$  so that  $V^d(s) = V_A(h_A, s) \leq V_B(h_B, s)$ , or  $p_A(h_A, s) = 1$  and  $V_A(h_A, s) < V_B(h_B, s)$ . The lemma is proved.  $\square$

With lemma 17 in hand, the proof of proposition 4 requires only one additional step. We need to show that the value function is uniquely determined by  $b$  and  $s$ . If two governments start with the same levels of  $b$  and  $s$ , either one of them can mimic the other but choose  $\epsilon$  less debt in the next period; lemma 17 implies that from this point forward, the mimicking government is weakly better off. The utility loss from paying down  $\epsilon$  debt in the initial period can be made arbitrarily small by choosing arbitrarily small  $\epsilon$ , and hence the mimicking government's value must be weakly higher. Since this argument works in both directions, we conclude that the value is indeed uniquely determined by  $b$  and  $s$ .

*Proof of proposition 4.* Consider  $(h_A, h_B)$  such that  $b(h_A) = b(h_B) = b$ . At  $(h_B, s)$  consider the feasible choice

$$\tilde{b}'_B = b'_A(h_A, s) - \epsilon$$

for some  $\epsilon > 0$ . Define continuation histories

$$\begin{aligned} h'_A &= \left( h_A, s, b'_A(h_A, s), \{q_A^i\} \right) \\ \tilde{h}'_B &= \left( h_B, s, \tilde{b}'_B, \{\tilde{q}_B^i\} \right) \end{aligned}$$

From Lemma 17,

$$V_B(\tilde{h}'_B, s') \geq V_A(h'_A, s') \quad (33)$$

and  $B$  repays if  $A$  repays. Hence  $\tilde{q}_B \geq q_A$  by the pricing condition (27). Moreover,

$$\begin{aligned} \tilde{c}_B &= c_A + \tilde{q}_B \tilde{b}'_B - q_A b'_A \\ &= c_A + (\tilde{q}_B - q_A) b'_A + \tilde{q}_B (\tilde{b}'_B - b'_A) \\ &\geq c_A - \frac{1}{R} \epsilon \end{aligned} \quad (34)$$

where the inequality follows as in (32) in the proof of Lemma 17.

Now,

$$\begin{aligned} V_B(h_B, s) - V_A(h_A, s) &\geq u(\tilde{c}_B) - u(c_A) + \beta \sum_{s'} \left( V_B(\tilde{h}'_B, s') - V_A(h'_A, s') \right) \\ &\geq u\left(c_A - \frac{1}{R} \epsilon\right) - u(c_A) \end{aligned}$$

where inequality follows from (33) and (34). Taking the limit as  $\epsilon \rightarrow 0$  and using continuity of  $u$ , we obtain  $V_B(h_B, s) \geq V_A(h_A, s)$ . The symmetric argument implies that  $V_B(h_B, s) \leq V_A(h_A, s)$ , which concludes the proof.  $\square$

### A.3 Proof of proposition 7 (Bulow-Rogoff)

*Proof.* We first verify that when  $V^d(s) = V^{nb}(0, s)$ , there exists an equilibrium where the government will default for any positive amount of debt  $b > 0$ . This equilibrium is  $(V^{nb}, Q^{nb})$ , where  $V^{nb}$  is given in (5) and the government faces

$$Q^{nb}(b', s) = \begin{cases} \frac{b'}{R} & b' \leq 0 \\ 0 & b' \geq 0 \end{cases} \quad (35)$$

which is the revenue schedule induced by default thresholds identically equal to zero.

First,  $Q^{nb}$  generates  $V^{nb}$ . The budget constraint in (5) is effectively the same as the constraint in (2) given prices (35); although (5) does not allow  $b' > 0$  while (2) does, positive borrowing  $b' > 0$  will never be optimal given prices (35) because it raises no revenue. Moreover, proposition 1 shows that the value function generated by the prices in (35) is decreasing in  $b$ ; hence whenever  $b \leq 0$ , we have  $V^{nb}(b, s) \geq V^{nb}(0, s) = V^d(s)$  for all  $s$ , so that default is never optimal.

Second, the default thresholds corresponding to  $V^{nb}$  are identically equal to zero, thereby generating  $Q^{nb}$ . This also follows from the monotonicity of  $V^{nb}$  in  $b$  (proposition 1): since

$$V^{nb}(b, s) \geq V^d(s) = V^{nb}(0, s) \iff b \leq 0 \quad \forall s$$

we have  $b^*(s) = 0$  for all  $s$ .

Proposition 3 then implies that  $(V^{nb}, Q^{nb})$  must be the unique Markov perfect equilibrium, and hence that there is no distinct equilibrium in which debt can be sustained. In particular, there is no equilibrium where the expectation of being able to borrow in the future is enough to discourage default and sustain some positive debt. This is the incomplete markets version of the **Bulow and Rogoff (1989)** result.  $\square$

#### A.4 Proof of lemma 8

*Proof.* This equilibrium can be verified almost immediately. If  $b^*(s) = 0$  for all  $s$ , then  $Q(b', s) = 0$  for all  $s$  and feasible  $b'$  by (7). A decision rule  $b'(b, s) \equiv 0$ , where  $b'$  is identically equal to zero, is weakly optimal, since higher  $b'$  offers no revenue and lower  $b'$  is not feasible. It follows that  $V(0, s) = u(y(s), s) + \beta \mathbb{E}[V(0, s')|s']$ , and hence that  $V(0, s)$  equals  $V^{aut}(s) = V^d(s)$ . For  $b > 0$ , then,

$$V(b, s) = u(y(s) - b, s) + \beta \mathbb{E}[V^{aut}(s')|s'] < u(y(s), s) + \beta \mathbb{E}[V^{aut}(s')|s'] = V^{aut}(s) = V^d(s)$$

and it is indeed optimal to default whenever  $b > 0$ , verifying the equilibrium with debt thresholds  $b^*(s) = 0$ .  $\square$

#### A.5 Proof of proposition 9

*Proof.* Suppose first that the government can borrow at the risk-free rate  $R$ . Assuming that the lower bound  $\underline{b} = 0$  on  $b$  is not binding, then since  $\beta R = 1$  it is optimal to consume at a constant level  $c$  in all periods.<sup>17</sup>

If the government faces  $b = 0$  and  $s = s_L$ , then this level of consumption is

$$c = \frac{y(s_L) + \beta y(s_H)}{1 + \beta}$$

and to achieve it the government must borrow  $b'(0, s_L) = R(c - y(s_L)) = \frac{y(s_H) - y(s_L)}{1 + \beta}$ . Then, when the government enters the next period with this debt and  $s = s_H$ , its endowment  $y(s_H)$  is exactly enough to repay the debt and consume  $c$ :  $y(s_H) = c + b'(0, s_L)$ . It follows that the optimal plan

<sup>17</sup>Otherwise, if  $c_t < c_{t+1}$  for any  $t$ , then perturbing consumption to  $(c_t + \epsilon, c_{t+1} - R\epsilon)$  provides higher utility

$$v(c_t + \epsilon) + \beta v(c_{t+1} - R\epsilon) > v(c_t) + \beta v(c_{t+1})$$

due to strict concavity of  $v$  and  $\beta R = 1$ , and similarly for  $c_t > c_{t+1}$ .

involves alternating between  $(b, s) = (0, s_L)$  and  $(b, s) = \left(\frac{y(s_H) - y(s_L)}{1 + \beta}, s_H\right)$ .

Let us compare the value of this plan to the value of autarky/default, starting at  $s_H$ . We obtain

$$\begin{aligned}
& \frac{v(c)}{1 - \beta} - V^{aut}(s_H) \\
&= v(c) + \beta v(c) + \dots - v(y(s_H)) - \beta v(y(s_L)) - \dots \\
&= (v(c) - v(y(s_H))) + \beta (v(c) + \beta u(c) + \dots - v(y(s_L)) - \beta v(y(s_H)) - \dots) \\
&= (v(c) - v(y(s_H))) + \frac{\beta}{1 - \beta^2} (v(c) + \beta v(c) - v(y(s_L)) - \beta v(y(s_H))) \tag{36}
\end{aligned}$$

In (36), the first term in parentheses is strictly negative, while the second term in parentheses is strictly positive by strict concavity of  $v$ . Since  $\beta/(1 - \beta^2) \rightarrow \infty$  as  $\beta \rightarrow 1$ , however, for  $\beta$  sufficiently close to 1 the second term will always dominate the first, and  $v(c)/(1 - \beta)$  will exceed  $V^{aut}(s_H)$ . In this case, it is strictly suboptimal for the government to default at  $(b, s) = \left(\frac{y(s_H) - y(s_L)}{1 + \beta}, s_H\right)$ . It also follows that  $v(c)/(1 - \beta) > V^{aut}(s_H) > V^{aut}(s_L)$ , and therefore that it is also strictly suboptimal for the government to default at  $(b, s) = (0, s_L)$ .

We have shown that if the government can borrow at the risk-free rate  $R$  up to  $\frac{y(s_H) - y(s_L)}{1 + \beta}$  when in the low state, its optimal plan starting from either  $(b, s) = (0, s_L)$  or  $(b, s) = \left(\frac{y(s_H) - y(s_L)}{1 + \beta}, s_H\right)$  is to alternate between the two. In this case, it is strictly suboptimal to default at either point, and therefore creditors will indeed lend at rate  $R$  up to an amount strictly greater than  $\frac{y(s_H) - y(s_L)}{1 + \beta}$  in the low state.

To finish constructing the equilibrium, we need to find the thresholds  $b^*(s_L)$  and  $b^*(s_H)$  associated with that equilibrium. For this, we rely on the machinery developed in the existence proof in A.1.3. In terms of the  $\mathbb{T}$  operator defined in that proof, we have already shown that if  $b^{*0} = (b^{*0}(s_L), b^{*0}(s_H)) \equiv \left(0, \frac{y(s_H) - y(s_L)}{1 + \beta}\right)$ , then  $\mathbb{T}b^{*0} > b^{*0}$ . Iterating forward as in A.1.3 to obtain  $b^{*\infty} = \mathbb{T}^\infty b^{*0}$ , it follows from monotonicity of  $\mathbb{T}$  that  $b^{*\infty} > b^{*0} \geq 0$ , with these thresholds being part of a Markov perfect equilibrium. This finishes our construction.  $\square$

## A.6 Proof of lemma 10

*Proof.* First, note that for any  $x$  and  $s$ , we have

$$\begin{aligned}
\tilde{Q}(\lambda x + \underline{b}, s) &= \frac{(\lambda x + \underline{b})}{R} \sum_{\{s': \lambda x \leq \tilde{b}^*(s') - \underline{b}\}} \pi(s'|s) \\
&\geq (1 - \lambda)\underline{b} + \frac{\lambda(x + \underline{b})}{R} \sum_{\{s': x \leq \tilde{b}^*(s') - \underline{b}\}} \pi(s'|s) = \lambda Q(x + \underline{b}, s) + (1 - \lambda)\underline{b} \tag{37}
\end{aligned}$$

where there is strict inequality if  $\underline{b} < 0$ .

Now we can formally define the *mimicking at a distance* policy. Suppose that at time 0 we have state  $s$  and debt level  $b$ . The equilibrium  $(V, Q)$  induces an allocation  $\{c(s^t), b(s^{t-1}), p(s^t)\}_{s^t \succeq s^0}$  at all histories following  $s^0$ . We construct a policy for the government in the equilibrium  $(\tilde{V}, \tilde{Q})$

starting at  $s^0$  as follows. For every history  $s^t \succeq s^0$ , let  $\tilde{p}(s^t) = p(s^t)$ , and whenever  $p(s^t) = 1$  define a plan for debt

$$\tilde{b}(s^{t-1}) - \underline{b} = \lambda(b(s^{t-1}) - \underline{b})$$

The resulting consumption path, again for  $p(s^t) = 1$ , satisfies

$$\begin{aligned} \tilde{c}(s^t) &= y(s_t) - \tilde{b}(s^{t-1}) + \tilde{Q}(\tilde{b}(s^t), s_t) \\ &= y(s_t) - \lambda b(s^{t-1}) - (1 - \lambda)\underline{b} + Q(\lambda(b(s^t) - \underline{b}) + \underline{b}, s_t) \\ &\geq y(s_t) - \lambda b(s^{t-1}) - (1 - \lambda)\underline{b} + \lambda Q(b(s^t), s_t) + (1 - \lambda)\underline{b} \\ &= (1 - \lambda)y(s_t) + \lambda(y(s_t) - b(s^{t-1}) + Q(b(s^t), s_t)) \\ &= (1 - \lambda)y(s_t) + \lambda c(s^t) \end{aligned}$$

Using the concavity of  $u$ , whenever  $p(s^t) = 1$  we have

$$u(\tilde{c}(s^t), s_t) \geq (1 - \lambda)u(y(s_t), s_t) + \lambda u(c(s^t), s_t) \geq (1 - \lambda)v^d(s_t) + \lambda u(c(s^t), s_t) \quad (38)$$

where the strict inequality from (37) persists in (38) whenever  $\underline{b} < 0$ , and by assumption,  $u(y(s_t), s_t) > v^d(s_t)$  gives strict inequality whenever  $\underline{b} = 0$ . Summing (38) across all times and states where  $p(s^t) = 1$ , together with  $v^d(s_t) = v^d(s_t)$  across all times and states where  $p(s^t) = 0$ , we obtain the result

$$\tilde{V}(\tilde{b}, s) > (1 - \lambda)V^d(s) + \lambda V(b, s)$$

□

## A.7 Proof of proposition 12

*Proof.* Suppose to the contrary that there exist two distinct equilibria  $(V, Q)$  and  $(\tilde{V}, \tilde{Q})$ , with associated default thresholds  $\{b^*(s)\}_{s \in \mathcal{S}}$  and  $\{\tilde{b}^*(s)\}_{s \in \mathcal{S}}$ , such that  $Q(b, s) \geq \tilde{Q}(b, s)$  for all  $b$  and  $s$ . It follows that  $V(b, s) \geq \tilde{V}(b, s)$  for all  $b$  and  $s$  as well, since a government facing the weakly higher revenue schedule  $Q$  can always replicate the policy of the government facing  $\tilde{Q}$ , achieving weakly higher consumption in the process.<sup>18</sup>

Since  $Q$  and  $\tilde{Q}$  are distinct, there exists some  $s'$  such that  $b^*(s') > \tilde{b}^*(s')$ , and we define

$$M = \max_s b^*(s) - \tilde{b}^*(s) > 0 \quad (39)$$

We first seek to prove that, for any  $s$  and  $b \leq b^*(s)$

$$\tilde{V}(b - M, s) - \tilde{V}^d(s) > V(b, s) - V^d(s) \quad (40)$$

<sup>18</sup>Explicitly, it can set  $b = \tilde{b}$ ,  $p = \tilde{p}$ ,  $c = \tilde{c} + Q(b, s) - \tilde{Q}(b, s) \geq 0$ .



To do so, we use the same mimicking at a distance argument as in lemma 2, although the calculation becomes somewhat more complicated. Writing  $s^0 \equiv s$ , we continue to set  $\tilde{b}(s^t) = b(s^t) - M$  and  $\tilde{p}(s^t) = p(s^t)$ , along with the consumption policy  $\tilde{c}(s^t)$  in (11). This strategy places a lower bound on  $\tilde{V}(b - M, s)$ :

$$\tilde{V}(b - M, s) \geq \sum_{p(s^t)=1} \beta^t \Pi(s^t) u(\tilde{c}(s^t)) + \sum_{p(s^t)=0, p(s^{t-1})=1} \beta^t \Pi(s^t) \tilde{V}^d(s_t) \quad (41)$$

Subtracting the corresponding expression for  $V(b, s)$ , and using  $\tilde{c}(s^t) > c(s^t)$ , we have

$$\tilde{V}(b - M, s) - V(b, s) > \sum_{p(s^t)=0, p(s^{t-1})=1} \beta^t \Pi(s^t) \left( \tilde{V}^d(s_t) - V^d(s_t) \right) \quad (42)$$

Subtracting  $\tilde{V}^d(s) - V^d(s)$  from both sides we obtain

$$\begin{aligned} & \left( \tilde{V}(b - M, s) - \tilde{V}^d(s) \right) - \left( V(b, s) - V^d(s) \right) \\ & > -(\tilde{V}^d(s) - V^d(s)) + \sum_{p(s^t)=0, p(s^{t-1})=1} \beta^t \Pi(s^t) \left( \tilde{V}^d(s_t) - V^d(s_t) \right) \end{aligned} \quad (43)$$

and to prove (40) it suffices to show that the right side of (43) is nonnegative.

Expanding  $\tilde{V}^d(s_t) - V^d(s_t)$  gives

$$\tilde{V}^d(s_t) - V^d(s_t) = \sum_{s_\tau} \beta^{\tau-t} (1 - \lambda) \lambda^{\tau-t-1} \Pi(s_\tau | s_t) \left( \tilde{V}^o(0, s_\tau) - V^o(0, s_\tau) \right) \quad (44)$$

Now, using (44), we can rewrite the right side of (42) as

$$\begin{aligned} & - \sum_{s^\tau \succ s^0} \beta^\tau (1 - \lambda) \lambda^{\tau-1} \Pi(s^\tau) \left( \tilde{V}^o(0, s_\tau) - V^o(0, s_\tau) \right) \\ & + \sum_{p(s^t)=0, p(s^{t-1})=1} \sum_{s^\tau \succ s^t} \beta^\tau (1 - \lambda) \lambda^{\tau-t-1} \Pi(s^\tau) \left( \tilde{V}^o(0, s_\tau) - V^o(0, s_\tau) \right) \end{aligned}$$

which can be rearranged as

$$\lambda^{-1} (1 - \lambda) \sum_{s^\tau \succ s^0} \beta^\tau \Pi(s^\tau) \left( V^o(0, s_\tau) - \tilde{V}^o(0, s_\tau) \right) \cdot \left( 1 - \sum_{s^\tau \succ s^t \succ s^0} \lambda^{\tau-t} \cdot \mathbf{1}_{\{p(s^t)=0, p(s^{t-1})=1\}} \right) \quad (45)$$

Since for any  $s^\tau$  there exists at most one  $s^t$  such that  $p(s^t) = 0$  and  $p(s^{t-1}) = 1$ , the rightmost factor in parentheses is nonnegative. Since in addition  $V(0, s_\tau) \geq \tilde{V}(0, s_\tau)$ , the preceding factor is nonnegative as well, and hence (45) is nonnegative. (40) therefore follows.

Finally, suppose that the maximum in (39) is attained at  $\bar{s}$ , so that  $b^*(\bar{s}) = \tilde{b}^*(\bar{s}) + M$ . Applying (40), we have

$$0 = \tilde{V}(\tilde{b}^*(\bar{s}), \bar{s}) - \tilde{V}^d(\bar{s}) > V(b^*(\bar{s}), \bar{s}) - V^d(\bar{s}) = 0$$

which is a contradiction.  $\square$

### A.8 Proof of proposition 13

*Proof.* Write  $V^{re} = \mathbb{E}_{s'}[V^o(0, s')]$ , and similarly  $\tilde{V}^{re}$  for a conjectured alternative equilibrium. First, observe that if  $\tilde{V}^{re} = V^{re}$ , then the two equilibria have the same expected value from default  $V^d$ , and we can apply proposition 3 taking  $V^d$  as given to conclude that the two equilibria must be the same.

Otherwise, assume without loss of generality that  $V^{re} > \tilde{V}^{re}$ . It cannot be that  $\tilde{Q}(b') \geq Q(b')$  for all  $b'$ , since in that case a government starting with zero debt and facing the weakly higher debt schedule  $\tilde{Q}$  could always replicate the policy of the government facing  $Q$ , achieving weakly higher consumption in the process. This would imply  $\tilde{V}^{re} \geq V^{re}$ , a contradiction. Hence  $Q(b') > \tilde{Q}(b')$  for some  $b'$ . From this point on, the proof is the same as the proof for proposition 12 starting with the definition of  $M$  in (39), except that we can replace (44) with simply

$$\tilde{V}^d(s_t) - V^d(s_t) = \sum_{\tau > t} \beta^{\tau-t} (1 - \lambda) \lambda^{\tau-t-1} (\tilde{V}^{re} - V^{re}) \quad (46)$$

allowing us to replace (45) with

$$\lambda^{-1} (1 - \lambda) \sum_{\tau > 0} \beta^\tau (V^{re} - \tilde{V}^{re}) \cdot \left( 1 - \sum_{\tau > t > 0} \lambda^{\tau-t} \cdot \mathbf{1}_{\{p(s^t)=0, p(s^{t-1})=1\}} \right) \quad (47)$$

again concluding that the expression is nonnegative, from which a contradiction follows.  $\square$